MEASURE and INTEGRATION
Problems with Solutions

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NOTATIONS

$\mathcal{A}(X)$: The σ-algebra of subsets of X.
$(X, \mathcal{A}(X), \mu)$: The measure space on X.
$\mathcal{B}(X)$: The σ-algebra of Borel sets in a topological space X.
$\mathcal{M}_L$: The σ-algebra of Lebesgue measurable sets in $\mathbb{R}$.
$(\mathbb{R}, \mathcal{M}_L, \mu_L)$: The Lebesgue measure space on $\mathbb{R}$.
μ_L: The Lebesgue measure on $\mathbb{R}$.
μ_L^*: The Lebesgue outer measure on $\mathbb{R}$.
$\mathbf{1}_E$ or χ_E: The characteristic function of the set E.

Contents

Chapter 1

Measure on a σ-Algebra of Sets

1. Limits of sequences of sets

Definition 1 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a set X.*
(a) We say that (A_n) is increasing if $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, and decreasing if $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$.
(b) For an increasing sequence (A_n), we define

$$\lim_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} A_n.$$

For a decreasing sequence (A_n), we define

$$\lim_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} A_n.$$

Definition 2 *For any sequence (A_n) of subsets of a set X, we define*

$$\liminf_{n\to\infty} A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$$

$$\limsup_{n\to\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k.$$

Proposition 1 *Let (A_n) be a sequence of subsets of a set X. Then*

$$(i) \quad \liminf_{n\to\infty} A_n = \{x \in X : \ x \in A_n \ \text{ for all but finitely many } \ n \in \mathbb{N}\}.$$

$$(ii) \quad \limsup_{n\to\infty} A_n = \{x \in X : \ x \in A_n \ \text{ for infinitely many } \ n \in \mathbb{N}\}.$$

$$(iii) \quad \liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.$$

2. σ-algebra of sets

Definition 3 *(σ-algebra)*
Let X be an arbitrary set. A collection $\mathcal{A}$ of subsets of X is called an algebra if it satisfies the following conditions:

1. $X \in \mathcal{A}$.
2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.
3. $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.
   An algebra $\mathcal{A}$ of a set X is called a σ-algebra if it satisfies the additional condition:
4. $A_n \in \mathcal{A},\ \forall n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in n \in \mathbb{N}$.

Definition 4 *(Borel σ-algebra)*
Let $(X, \mathcal{O})$ be a topological space. We call the Borel σ-algebra $\mathcal{B}(X)$ the smallest σ-algebra of X containing $\mathcal{O}$.

It is evident that open sets and closed sets in X are Borel sets.

3. Measure on a σ-algebra

Definition 5 *(Measure)*
Let $\mathcal{A}$ be a σ-algebra of subsets of X. A set function μ defined on $\mathcal{A}$ is called a measure if it satisfies the following conditions:

1. $\mu(E) \in [0, \infty]$ *for every* $E \in \mathcal{A}$.
2. $\mu(\varnothing) = 0$.
3. $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, *disjoint* $\Rightarrow \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$.

Notice that if $E \in \mathcal{A}$ such that $\mu(E) = 0$, then E is called a *null set*. If any subset E_0 of a null set E is also a null set, then the measure space $(X, \mathcal{A}, \mu)$ is called *complete*.

Proposition 2 *(Properties of a measure)*
A measure μ on a σ-algebra $\mathcal{A}$ of subsets of X has the following properties:
(1) Finite additivity: $(E_1, E_2, ..., E_n) \subset \mathcal{A}$, *disjoint* $\Longrightarrow \mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(E_k)$.
(2) Monotonicity: $E_1, E_2 \in \mathcal{A},\ E_1 \subset E_2 \Longrightarrow \mu(E_1) \leq m(E_2)$.
(3) $E_1, E_2 \in \mathcal{A},\ E_1 \subset E_2,\ \mu(E_1) < \infty \Longrightarrow \mu(E_2 \setminus E_1) = \mu(E_2) - \mu(E_1)$.
(4) Countable subadditivity: $(E_n) \subset \mathcal{A} \Longrightarrow \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu(E_n)$.

Definition 6 *(Finite, σ-finite measure)*
Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. μ *is called finite if* $\mu(X) < \infty$.
2. μ *is called σ-finite if there exists a sequence (E_n) of subsets of X such that*

$$X = \bigcup_{n\in\mathbb{N}} E_n \quad \text{and} \quad \mu(E_n) < \infty,\ \forall n \in \mathbb{N}.$$

4. Outer measures

Definition 7 *(Outer measure)*
Let X be a set. A set function μ^* defined on the σ-algebra $\mathcal{P}(X)$ of all subsets of X is called an outer measure on X if it satisfies the following conditions:

(i) $\mu^*(E) \in [0, \infty]$ *for every* $E \in \mathcal{P}(X)$.

(ii) $\mu^*(\emptyset) = 0$.

(iii) $E, F \in \mathfrak{P}(X),\ E \subset F \Rightarrow \mu^*(E) \leq \mu^*(F)$.

(iv) countable subadditivity:

$$(E_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X),\ \mu^*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(E_n).$$

Definition 8 *(Caratheodory condition)*
We say that $E \in \mathcal{P}(X)$ is μ^*-measurable if it satisfies the Caratheodory condition:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \ \text{ for every } \ A \in \mathcal{P}(X).$$

We write $\mathcal{M}(\mu^*)$ for the collection of all μ^*-measurable $E \in \mathcal{P}(X)$. Then $\mathcal{M}(\mu^*)$ is a σ-algebra.

Proposition 3 *(Properties of μ^*)*
(a) If $E_1, E_2 \in \mathcal{M}(\mu^*)$, then $E_1 \cup E_2 \in \mathcal{M}(\mu^*)$.
(b) μ^* is additive on $\mathcal{M}(\mu^*)$, that is,

$$E_1, E_2 \in \mathcal{M}(\mu^*),\ E_1 \cap E_2 = \varnothing \Longrightarrow \mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2).$$

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Problem 1
Let $\mathcal{A}$ be a collection of subsets of a set X with the following properties:

1. $X \in \mathcal{A}$.
2. $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$.

Show that $\mathcal{A}$ is an algebra.

Solution
(i) $X \in \mathcal{A}$.
(ii) $A \in \mathcal{A} \Rightarrow A^c = X \setminus A \in \mathcal{A}$ (by 2).
(iii) $A, B \in \mathcal{A} \Rightarrow A \cap B = A \setminus B^c \in \mathcal{A}$ since $B^c \in \mathcal{A}$ (by (ii)).
Since $A^c, B^c \in \mathcal{A}$, $(A \cup B)^c = A^c \cap B^c \in \mathcal{A}$. Thus, $A \cup B \in \mathcal{A}$. ■

Problem 2
(a) Show that if $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of algebras of subsets of a set X, then $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is an algebra of subsets of X.
(b) Show by example that even if $\mathcal{A}_n$ in (a) is a σ-algebra for every $n \in \mathbb{N}$, the union still may not be a σ-algebra.

Solution
(a) Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. We show that $\mathcal{A}$ is an algebra.

(i) Since $X \in \mathcal{A}_n$, $\forall n \in \mathbb{N}$, so $X \in \mathcal{A}$.

(ii) Let $A \in \mathcal{A}$. Then $A \in \mathcal{A}_n$ for some n. And so $A^c \in \mathcal{A}_n$ (since $\mathcal{A}_n$ is an algebra). Thus, $A^c \in \mathcal{A}$.

(iii) Suppose $A, B \in \mathcal{A}$. We shall show $A \cup B \in \mathcal{A}$.
Since $\{\mathcal{A}_n\}$ is increasing, i.e., $\mathcal{A}_1 \subset \mathcal{A}_2 \subset ...$ and $A, B \in \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$, there is some $n_0 \in \mathbb{N}$ such that $A, B \in \mathcal{A}_0$. Thus, $A \cup B \in \mathcal{A}_0$. Hence, $A \cup B \in \mathcal{A}$.

(b) Let $X = \mathbb{N}$, $\mathcal{A}_n =$ the family of all subsets of $\{1, 2, ..., n\}$ and their complements. Clearly, $\mathcal{A}_n$ is a σ-algebra and $\mathcal{A}_1 \subset \mathcal{A}_2 \subset ....$ However, $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is the family of all finite and co-finite subsets of $\mathbb{N}$, which is not a σ-algebra. ■

Problem 3
Let X be an arbitrary infinite set. We say that a subset A of X is co-finite if its complement A^c is a finite subset of X. Let $\mathcal{A}$ consists of all the finite and the co-finite subsets of a set X.
(a) Show that $\mathcal{A}$ is an algebra of subsets of X.
(b) Show that $\mathcal{A}$ is a σ-algebra if and only if X is a finite set.

Solution
(a)

(i) $X \in \mathcal{A}$ since X is co-finite.

(ii) Let $A \in \mathcal{A}$. If A is finite then A^c is co-finite, so $A^c \in \mathcal{A}$. If A co-finite then A^c is finite, so $A^c \in \mathcal{A}$. In both cases,

$$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}.$$

(iii) Let $A, B \in \mathcal{A}$. We shall show $A \cup B \in \mathcal{A}$.
If A and B are finite, then $A \cup B$ is finite, so $A \cup B \in \mathcal{A}$. Otherwise, assume that A is co-finite, then $A \cup B$ is co-finite, so $A \cup B \in \mathcal{A}$. In both cases,

$$A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}.$$

(b) If X is finite then $\mathcal{A} = \mathcal{P}(X)$, which is a σ-algebra.
To show the reserve, i.e., if $\mathcal{A}$ is a σ-algebra then X is finite, we assume that X is infinite. So we can find an infinite sequence $(a_1, a_2, ...)$ of distinct elements of X such that $X \setminus \{a_1, a_2, ...\}$ is infinite. Let $A_n = \{a_n\}$. Then $A_n \in \mathcal{A}$ for any $n \in \mathbb{N}$, while $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is neither finite nor co-finite. So $\bigcup_{n\in\mathbb{N}} A_n \notin \mathcal{A}$. Thus, $\mathcal{A}$ is not a σ-algebra: a contradiction! ■

Note:
For an arbitrary collection $\mathcal{C}$ of subsets of a set X, we write $\sigma(\mathcal{C})$ for the smallest σ-algebra of subsets of X containing $\mathcal{C}$ and call it the σ-algebra generated by $\mathcal{C}$.

Problem 4
Let $\mathcal{C}$ be an arbitrary collection of subsets of a set X. Show that for a given $A \in \sigma(\mathcal{C})$, there exists a countable sub-collection $\mathcal{C}_A$ of $\mathcal{C}$ depending on A such that $A \in \sigma(\mathcal{C}_A)$. (We say that every member of $\sigma(\mathcal{C})$ is countable generated).

Solution
Denote by $\mathcal{B}$ the family of all subsets A of X for which there exists a countable sub-collection $\mathcal{C}_A$ of $\mathcal{C}$ such that $A \in \sigma(\mathcal{C}_A)$. We claim that $\mathcal{B}$ is a σ-algebra and that $\mathcal{C} \subset \mathcal{B}$.
The second claim is clear, since $A \in \sigma(\{A\})$ for any $A \in \mathcal{C}$. To prove the first one, we have to verify that $\mathcal{B}$ satisfies the definition of a σ-algebra.

(i) Clearly, $X \in \mathcal{B}$.

(ii) If $A \in \mathcal{B}$ then $A \in \sigma(\mathcal{C}_A)$ for some countable family $\mathcal{C}_A \subset \sigma(\mathcal{C})$. Then $A^c \in \sigma(\mathcal{C}_A)$, so $A^c \in \mathcal{B}$.

(iii) Suppose $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{B}$. Then $A_n \in \sigma(\mathcal{C}_{A_n})$ for some countable family $\mathcal{C}_{A_n} \subset \mathcal{C}$. Let $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_{A_n}$ then $\mathcal{E}$ is countable and $\mathcal{E} \subset \mathcal{C}$ and $A_n \in \sigma(\mathcal{E})$ for all $n \in \mathbb{N}$. By definition of σ-algebra, $\bigcup_{n\in\mathbb{N}} A_n \in \sigma(\mathcal{E})$, and so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}$.

Thus, $\mathcal{B}$ is a σ-algebra of subsets of X and $\mathcal{E} \subset \mathcal{B}$. Hence,

$$\sigma(\mathcal{E}) \subset \mathcal{B}.$$

By definition of $\mathcal{B}$, this implies that for every $A \in \sigma(\mathcal{C})$ there exists a countable $\mathcal{E} \subset \mathcal{C}$ such that $A \in \sigma(\mathcal{E})$. ■

Problem 5
Let γ a set function defined on a σ-algebra $\mathcal{A}$ of subsets of X. Show that it γ is additive and countably subadditive on $\mathcal{A}$, then it is countably additive on $\mathcal{A}$.

Solution
We first show that the additivity of γ implies its monotonicity. Indeed, let $A, B \in \mathcal{A}$ with $A \subset B$. Then

$$B = A \cup (B \setminus A) \quad \text{and} \quad A \cap (B \setminus A) = \varnothing.$$

Since γ is additive, we get

$$\gamma(B) = \gamma(A) + \gamma(B \setminus A) \geq \gamma(A).$$

Now let (E_n) be a disjoint sequence in $\mathcal{A}$. For every $N \in \mathbb{N}$, by the monotonicity and the additivity of γ, we have

$$\gamma\left(\bigcup_{n\in\mathbb{N}} E_n\right) \geq \gamma\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} \gamma(E_n).$$

Since this holds for every $N \in \mathbb{N}$, so we have

$$(i) \qquad \gamma\left(\bigcup_{n\in\mathbb{N}} E_n\right) \geq \sum_{n\in\mathbb{N}} \gamma(E_n).$$

On the other hand, by the countable subadditivity of γ, we have

$$(ii) \qquad \gamma\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \gamma(E_n).$$

From (i) and (ii), it follows that

$$\gamma\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \gamma(E_n).$$

This proves the countable additivity of γ. ■

Problem 6
Let X be an infinite set and $\mathcal{A}$ be the algebra consisting of the finite and co-finite subsets of X (cf. Prob.3). Define a set function μ on $\mathcal{A}$ by setting for every $A \in \mathcal{A}$:

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ 1 & \text{if } A \text{ is co-finite.} \end{cases}$$

(a) Show that μ is additive.
(b) Show that when X is countably infinite, μ is not additive.
(c) Show that when X is countably infinite, then X is the limit of an increasing sequence $\{A_n : n \in \mathbb{N}\}$ in $\mathcal{A}$ with $\mu(A_n) = 0$ for every $n \in \mathbb{N}$, but $\mu(X) = 1$.
(d) Show that when X is uncountably, the μ is countably additive.

Solution
(a) Suppose $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$ (i.e., $A \subset B^c$ and $B \subset A^c$).
If A is co-finite then B is finite (since $B \subset A^c$). So $A \cup B$ is co-finite. We have $\mu(A \cup B) = 1$, $\mu(A) = 1$ and $\mu(B) = 0$. Hence, $\mu(A \cup B) = \mu(A) + \mu(B)$.
If B is co-finite then A is finite (since $A \subset B^c$). So $A \cup B$ is co-finite, and we have the same result. Thus, μ is additive.
(b) Suppose X is countably infinite. We can then put X under this form: $X = \{x_1, x_2, ...\}$, $x_i \neq x_j$ if $i \neq j$. Let $A_n = \{x_n\}$. Then the family $\{A_n\}_{n\in\mathbb{N}}$ is disjoint and $\mu(A_n) = 0$ for every $n \in \mathbb{N}$. So $\sum_{n\in\mathbb{N}} \mu(A_n) = 0$. On the other hand, we have

$\bigcup_{n\in\mathbb{N}} A_n = X$, and $\mu(X) = 1$. Thus,

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \neq \sum_{n\in\mathbb{N}} \mu(A_n).$$

Hence, μ is not additive.
(c) Suppose X is countably infinite, and $X = \{x_1, x_2, ...\}$, $x_i \neq x_j$ if $i \neq j$ as in (b). Let $B_n = \{x_1, x_2, ..., x_n\}$. Then $\mu(B_n) = 0$ for every $n \in \mathbb{N}$, and the sequence $(B_n)_{n\in\mathbb{N}}$ is increasing. Moreover,

$$\lim_{n\to\infty} B_n = \bigcup_{n\in\mathbb{N}} B_n = X \quad \text{and} \quad \mu(X) = 1.$$

(d) Suppose X is uncountably. Consider the family of disjoint sets $\{C_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$. Suppose $C = \bigcup_{n\in\mathbb{N}} C_n \in \mathcal{A}$. We first claim: At most one of the C_n's can be co-finite. Indeed, assume there are two elements C_n and C_m of the family are co-finite. Since $C_m \subset C_n^c$, so C_m must be finite: a contradiction.
Suppose C_{n_0} is the co-finite set. Then since $C \supset C_{n_0}$, C is also co-finite. Therefore,

$$\mu(C) = \mu\left(\bigcup_{n\in\mathbb{N}} C_n\right) = 1.$$

On the other hand, we have

$$\mu(C_{n_0}) = 1 \quad \text{and} \quad \mu(C_n) = 0 \quad \text{for} \quad n \neq n_0.$$

Thus,

$$\mu\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \sum_{n\in\mathbb{N}} \mu(C_n).$$

If all C_n are finite then $\bigcup_{n\in\mathbb{N}} C_n$ is finite, so we have

$$0 = \mu\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \sum_{n\in\mathbb{N}} \mu(C_n). \qquad \blacksquare$$

Problem 7
Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that for any $A, B \in \mathcal{A}$, we have the equality:

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

Solution
If $\mu(A) = \infty$ or $\mu(B) = \infty$, then the equality is clear. Suppose $\mu(A)$ and $\mu(B)$ are finite. We have

$$\begin{aligned} A \cup B &= (A \setminus B) \cup (A \cap B) \cup (B \setminus A), \\ A &= (A \setminus B) \cup (A \cap B) \\ B &= (B \setminus A) \cup (A \cap B). \end{aligned}$$

Notice that in these decompositions, sets are disjoint. So we have

$$\mu(A \cup B) = \mu(A \setminus B) + \mu(A \cap B) + \mu(B \setminus A), \tag{1.1}$$

$$\mu(A) + \mu(B) = 2\mu(A \cap B) + \mu(A \setminus B) + \mu(B \setminus A). \tag{1.2}$$

From (1.1) and (1.2) we obtain

$$\mu(A \cup B) - \mu(A) - \mu(B) = -\mu(A \cap B).$$

The equality is proved. ■

Problem 8
The symmetry difference of $A, B \in \mathcal{P}(X)$ is defined by

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

(a) Prove that

$$\forall A, B, C \in \mathcal{P}(X),\ A \triangle B \subset (A \triangle C) \cup (C \triangle B).$$

(b) Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that

$$\forall A, B, C \in \mathcal{A},\ \mu(A \triangle B) \leq \mu(A \triangle C) + \mu(C \triangle B).$$

Solution
(a) Let $x \in A \triangle B$. Suppose $x \in A \setminus B$. If $x \in C$ then $x \in C \setminus B$ so $x \in C \triangle B$. If $x \notin C$, then $x \in A \setminus C$, so $x \in A \triangle C$. In both cases, we have

$$x \in A \triangle B \Rightarrow x \in (A \triangle C) \cup (C \triangle B).$$

The case $x \in B \setminus A$ is dealt with the same way.
(b) Use subadditivity of μ and (a). ■

Problem 9
Let X be an infinite set and μ the counting measure on the σ-algebra $\mathcal{A} = \mathcal{P}(X)$. Show that there exists a decreasing sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that

$$\lim_{n\to\infty} E_n = \varnothing \quad \textit{with} \quad \lim_{n\to\infty} \mu(E_n) \neq 0.$$

Solution
Since X is a infinite set, we can find an countably infinite set $\{x_1, x_2, ...\} \subset X$ with $x_i \neq x_j$ if $i \neq j$. Let $E_n = \{x_n, x_{n+1}, ...\}$. Then $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{A}$ with

$$\lim_{n\to\infty} E_n = \varnothing \quad \text{and} \quad \lim_{n\to\infty} \mu(E_n) = 0. \quad \blacksquare$$

Problem 10 (Monotone sequence of measurable sets)
Let $(X, \mathcal{A}, \mu)$ be a measure space, and (E_n) be a monotone sequence in $\mathcal{A}$.
(a) If (E_n) is increasing, show that

$$\lim_{n\to\infty} \mu(E_n) = \mu\big(\lim_{n\to\infty} E_n\big).$$

(b) If (E_n) is decreasing, show that

$$\lim_{n\to\infty} \mu(E_n) = \mu\big(\lim_{n\to\infty} E_n\big),$$

provided that there is a set $A \in \mathcal{A}$ satisfying $\mu(A) < \infty$ and $A \supset E_1$.

Solution
Recall that if (E_n) is increasing then $\lim_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$, and if (E_n) is decreasing then $\lim_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} E_n \in \mathcal{A}$. Note also that if (E_n) is a monotone sequence in $\mathcal{A}$, then $\big(\mu(E_n)\big)$ is a monotone sequence in $[0, \infty]$ by the monotonicity of μ, so that $\lim_{n\to\infty} \mu(E_n)$ exists in $[0, \infty]$.

(a) Suppose (E_n) is increasing. Then the sequence $\big(\mu(E_n)\big)$ is also increasing. Consider the first case where $\mu(E_{n_0}) = \infty$ for some E_{n_0}. In this case we have $\lim_{n\to\infty} \mu(E_n) = \infty$. On the other hand,

$$E_{n_0} \subset \bigcup_{n\in\mathbb{N}} E_n = \lim_{n\to\infty} E_n \Longrightarrow \mu\big(\lim_{n\to\infty} E_n\big) \geq \mu(E_{n_0}) = \infty.$$

Thus

$$\mu\big(\lim_{n\to\infty} E_n\big) = \infty = \lim_{n\to\infty} \mu(E_n).$$

Consider the next case where $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Let $E_0 = \varnothing$, then consider the disjoint sequence (F_n) in $\mathcal{A}$ defined by $F_n = E_n \setminus E_{n-1}$ for all $n \in \mathbb{N}$. It is evident that

$$\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n.$$

Then we have

$$\begin{aligned}
\mu\big(\lim_{n\to\infty} E_n\big) = \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) &= \mu\left(\bigcup_{n\in\mathbb{N}} F_n\right) \\
&= \sum_{n\in\mathbb{N}} \mu(F_n) = \sum_{n\in\mathbb{N}} \mu(E_n \setminus E_{n-1}) \\
&= \sum_{n\in\mathbb{N}} \big[\mu(E_n) - \mu(E_{n-1})\big] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \big[\mu(E_k) - \mu(E_{k-1})\big] \\
&= \lim_{n\to\infty} \big[\mu(E_n) - \mu(E_0)\big] = \lim_{n\to\infty} \mu(E_n). \quad \square
\end{aligned}$$

(b) Suppose (E_n) is decreasing and assume the existence of a containing set A with finite measure. Define a disjoint sequence (G_n) in $\mathcal{A}$ by setting $G_n = E_n \setminus E_{n+1}$ for all $n \in \mathbb{N}$. We claim that

$$(1) \qquad E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} G_n.$$

To show this, let $x \in E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n$. Then $x \in E_1$ and $x \notin \bigcap_{n\in\mathbb{N}} E_n$. Since the sequence (E_n) is decreasing, there exists the first set E_{n_0+1} in the sequence not containing x. Then

$$x \in E_{n_0} \setminus E_{n_0+1} = G_{n_0} \Longrightarrow x \in \bigcup_{n\in\mathbb{N}} G_n.$$

Conversely, if $x \in \bigcup_{n\in\mathbb{N}} G_n$, then $x \in G_{n_0} = E_{n_0} \setminus E_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Now $x \in E_{n_0} \subset E_1$. Since $x \notin E_{n_0+1}$, we have $x \notin \bigcap_{n\in\mathbb{N}} E_n$. Thus $x \in E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n$. Hence (1) is proved.
Now by (1) we have

$$(2) \qquad \mu\left(E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} G_n\right).$$

Since $\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) \le \mu(E_1) \le \mu(A) < \infty$, we have

$$(3) \quad \mu\left(E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n\right) = \mu(E_1) - \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \mu(E_1) - \mu(\lim_{n\to\infty} E_n).$$

By the countable additivity of μ, we have

$$\begin{aligned}(4) \quad \mu\left(\bigcup_{n\in\mathbb{N}} G_n\right) &= \sum_{n\in\mathbb{N}} \mu(G_n) = \sum_{n\in\mathbb{N}} \mu(E_n \setminus E_{n+1}) \\ &= \sum_{n\in\mathbb{N}} \big[\mu(E_n) - \mu(E_{n+1})\big] \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \big[\mu(E_k) - \mu(E_{k+1})\big] \\ &= \lim_{n\to\infty} \big[\mu(E_1) - \mu(E_{n+1})\big] \\ &= \mu(E_1) - \lim_{n\to\infty} \mu(E_{n+1}).\end{aligned}$$

Substituting (3) and (4) in (2), we have

$$\mu(E_1) - \mu(\lim_{n\to\infty} E_n) = \mu(E_1) - \lim_{n\to\infty} \mu(E_{n+1}).$$

Since $\mu(E_1) < \infty$, we have

$$\mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(E_n). \qquad \blacksquare$$

Problem 11 (Fatou's lemma for μ)
Let $(X, \mathcal{A}, \mu)$ be a measure space, and (E_n) be a sequence in $\mathcal{A}$.
(a) Show that

$$\mu\big(\liminf_{n\to\infty} E_n\big) \leq \liminf_{n\to\infty} \mu(E_n).$$

(b) If there exists $A \in \mathcal{A}$ with $E_n \subset A$ and $\mu(A) < \infty$ for every $n \in \mathbb{N}$, then show that

$$\mu\big(\limsup_{n\to\infty} E_n\big) \geq \limsup_{n\to\infty} \mu(E_n).$$

Solution
(a) Recall that

$$\liminf_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} E_k = \lim_{n\to\infty} \bigcap_{k\geq n} E_k,$$

by the fact that $\left(\bigcap_{k\geq n} E_k\right)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{A}$. Then by Problem 9a we have

$$(*) \qquad \mu\big(\liminf_{n\to\infty} E_n\big) = \lim_{n\to\infty} \mu\left(\bigcap_{k\geq n} E_k\right) = \liminf_{n\to\infty} \mu\left(\bigcap_{k\geq n} E_k\right),$$

since the limit of a sequence, if it exists, is equal to the limit inferior of the sequence. Since $\bigcap_{k\geq n} E_k \subset E_n$, we have $\mu\left(\bigcap_{k\geq n} E_k\right) \leq \mu(E_n)$ for every $n \in \mathbb{N}$. This implies that

$$\liminf_{n\to\infty} \mu\left(\bigcap_{k\geq n} E_k\right) \leq \liminf_{n\to\infty} \mu(E_n).$$

Thus by $(*)$ we obtain

$$\mu\big(\liminf_{n\to\infty} E_n\big) \leq \liminf_{n\to\infty} \mu(E_n).$$

(b) Now

$$\limsup_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k = \lim_{n\to\infty} \bigcup_{k\geq n} E_k,$$

by the fact that $\left(\bigcup_{k\geq n} E_k\right)_{n\in\mathbb{N}}$ is an decreasing sequence in $\mathcal{A}$. Since $E_n \subset A$ for all $n \in \mathbb{N}$, we have $\bigcup_{k\geq n} E_k \subset A$ for all $n \in \mathbb{N}$. Thus by Problem 9b we have

$$\mu\big(\limsup_{n\to\infty} E_n\big) = \mu\left(\lim_{n\to\infty} \bigcup_{k\geq n} E_k\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k\geq n} E_k\right).$$

Now

$$\lim_{n\to\infty} \mu\left(\bigcup_{k\geq n} E_k\right) = \limsup_{n\to\infty} \mu\left(\bigcup_{k\geq n} E_k\right),$$

since the limit of a sequence, if it exists, is equal to the limit superior of the sequence. Then by $\bigcup_{k\geq n} E_k \supset E_n$ we have

$$\mu\left(\bigcup_{k\geq n} E_k\right) \geq \mu(E_n).$$

Thus

$$\limsup_{n\to\infty} \mu\left(\bigcup_{k\geq n} E_k\right) \geq \limsup_{n\to\infty} \mu(E_n).$$

It follows that

$$\mu\big(\limsup_{n\to\infty} E_n\big) \geq \limsup_{n\to\infty} \mu(E_n). \qquad \blacksquare$$

Problem 12
Let μ^* be an outer measure on a set X. Show that the following two conditions are equivalent:
(i) μ^* is additive on $\mathcal{P}(X)$.
(ii) Every element of $\mathcal{P}(X)$ is μ^*-measurable, that is, $\mathcal{M}(\mu^*) = \mathcal{P}(X)$.

Solution

• Suppose μ^* is additive on $\mathcal{P}(X)$. Let $E \in \mathcal{P}(X)$. Then for any $A \in \mathcal{P}(X)$,

$$A = (A\cap E) \cup (A \cap E^c) \text{ and } (A\cap E)\cap(A\cap E^c) = \varnothing.$$

By the additivity of μ^* on $\mathcal{P}(X)$, we have

$$\mu^*(A) = \mu^*(A\cap E) + \mu^*(A\cap E^c).$$

This show that E satisfies the Carathéodory condition. Hence $E \in \mathcal{M}(\mu^*)$. So $\mathcal{P}(X) \subset \mathcal{M}(\mu^*)$. But by definition, $\mathcal{M}(\mu^*) \subset \mathcal{P}(X)$. Thus

$$\mathcal{M}(\mu^*) = \mathcal{P}(X).$$

• Conversely, suppose $\mathcal{M}(\mu^*) = \mathcal{P}(X)$. Since μ^* is additive on $\mathcal{M}(\mu^*)$ by Proposition 3, so μ^* is additive on $\mathcal{P}(X)$. $\blacksquare$

Problem 13
Let μ^* be an outer measure on a set X.
(a) Show that the restriction μ of μ^* on the σ-algebra $\mathcal{M}(\mu^*)$ is a measure on $\mathcal{M}(\mu^*)$.
(b) Show that if μ^* is additive on $\mathcal{P}(X)$, then it is countably additive on $\mathcal{P}(X)$.

Solution
(a) By definition, μ^* is countably subadditive on $\mathcal{P}(X)$. Its restriction μ on $\mathcal{M}(\mu^*)$ is countably subadditive on $\mathcal{M}(\mu^*)$. By Proposition 3b, μ^* is additive on $\mathcal{M}(\mu^*)$. Therefore, by Problem 5, μ^* is countably additive on $\mathcal{M}(\mu^*)$. Thus, μ^* is a measure on $\mathcal{M}(\mu^*)$. But μ is the restriction of μ^* on $\mathcal{M}(\mu^*)$, so we can say that μ is a measure on $\mathcal{M}(\mu^*)$.

(b) If μ^* is additive on $\mathcal{P}(X)$, then by Problem 11, $\mathcal{M}(\mu^*) = \mathcal{P}(X)$. So μ^* is a measure on $\mathcal{P}(X)$ (Problem 5). In particular, μ^* is countably additive on $\mathcal{P}(X)$. ■

Chapter 2

Lebesgue Measure on $\mathbb{R}$

1. Lebesgue outer measure on $\mathbb{R}$

Definition 9 *(Outer measure)*
Lebesgue outer measure on $\mathbb{R}$ is a set function $\mu_L^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined by

$$\mu_L^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : \ A \subset \bigcup_{k=1}^{\infty} I_k, \ I_k \ \text{ is open interval in } \ \mathbb{R} \right\}.$$

Proposition 4 *(Properties of μ_L^*)*

1. $\mu_L^*(A) = 0$ *if A is at most countable.*
2. *Monotonicity:* $A \subset B \Rightarrow \mu_L^*(A) \leq \mu_L^*(B)$.
3. *Translation invariant:* $\mu_L^*(A + x) = \mu_L^*(A), \ \forall x \in \mathbb{R}$.
4. *Countable subadditivity:* $\mu_L^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu_L^*(A_n)$.
5. *Null set:* $\mu_L^*(A) = 0 \Rightarrow \mu_L^*(A \cup B) = \mu_L^*(B)$ *and* $\mu_L^*(B \setminus A) = \mu_L^*(B)$ *for all* $B \in \mathcal{P}(\mathbb{R})$.
6. *For any interval* $I \subset \mathbb{R}$, $\mu_L^*(I) = \ell(I)$.
7. *Regularity:*

$$\forall E \in \mathcal{P}(\mathbb{R}), \ \varepsilon > 0, \ \exists O \ \text{ open set in } \ \mathbb{R} : \ \ O \supset E \ \text{ and } \ \mu_L^*(E) \leq \mu_L^*(O) \leq \mu_L^*(E) + \varepsilon.$$

2. Measurable sets and Lebesgue measure on $\mathbb{R}$

Definition 10 *(Carathéodory condition)*
A set $E \subset \mathbb{R}$ is said to be Lebesgue measurable (or μ_L-measurable, or measurable) if, for all $A \subset \mathbb{R}$, we have

$$\mu_L^*(A) = \mu_L^*(A \cap E) + \mu_L^*(A \cap E^c).$$

Since μ_L^* is subadditive, the sufficient condition for Carathéodory condition is

$$\mu_L^*(A) \geq \mu_L^*(A \cap E) + \mu_L^*(A \cap E^c).$$

The family of all measurable sets is denoted by $\mathcal{M}_L$. We can see that $\mathcal{M}_L$ is a σ-algebra. The restriction of μ_L^* on $\mathcal{M}_L$ is denoted by μ_L and is called Lebesgue measure.

Proposition 5 *(Properties of μ_L)*

1. $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ *is a complete measure space.*
2. $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ *is σ-finite measure space.*
3. $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_L$, *that is, every Borel set is measurable.*
4. $\mu_L(O) > 0$ *for every nonempty open set in* $\mathbb{R}$.
5. $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ *is translation invariant.*
6. $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ *is positively homogeneous, that is,*

$$\mu_L(\alpha E) = |\alpha| \mu_L(E), \ \forall \alpha \in \mathbb{R}, \ E \in \mathcal{M}_L.$$

Note on F_σ and G_δ sets:
Let $(X, \mathcal{T})$ be a topological space.
- A subset E of X is called a F_σ-set if it is the union of countably many closed sets.
- A subset E of X is called a G_δ-set if it is the intersection of countably many open sets.
- If E is a G_δ-set then E^c is a F_σ-set and *vice versa*. Every G_δ-set is Borel set, so is every F_σ-set.

* * **

Problem 14
If E is a null set in $(\mathbb{R}, \mathcal{M}_L, \mu_L)$, prove that E^c is dense in $\mathbb{R}$.

Solution

For every open interval I in $\mathbb{R}$, $\mu_L(I) > 0$ (property of Lebesgue measure). If $\mu_L(E) = 0$, then by the monotonicity of μ_L, E cannot contain any open interval as a subset. This implies that

$$E^c \cap I = \varnothing$$

for any open interval I in $\mathbb{R}$. Thus E^c is dense in $\mathbb{R}$. ■

Problem 15
Prove that for every $E \subset \mathbb{R}$, there exists a G_δ-set $G \subset \mathbb{R}$ such that

$$G \supset E \quad \textit{and} \quad \mu_L^*(G) = \mu_L^*(E).$$

Solution
We use the regularity property of μ_L^* (Property 7).
For $\varepsilon = \frac{1}{n},\ n \in \mathbb{N}$, there exists an open set $O_n \subset \mathbb{R}$ such that

$$O_n \supset E \ \text{ and } \ \mu_L^*(E) \leq \mu_L^*(O_n) \leq \mu_L^*(E) + \frac{1}{n}.$$

Let $G = \bigcap_{n\in\mathbb{N}} O_n$. Then G is a G_δ-set and $G \supset E$. Since $G \subset O_n$ for every $n \in \mathbb{N}$, we have

$$\mu_L^*(E) \leq \mu_L^*(G) \leq \mu_L^*(O_n) \leq \mu_L^*(E) + \frac{1}{n}.$$

This holds for every $n \in \mathbb{N}$, so we have

$$\mu_L^*(E) \leq \mu_L^*(G) \leq \mu_L^*(E).$$

Therefore

$$\mu^*(G) = \mu^*(E). \qquad \blacksquare$$

Problem 16
Let $E \subset \mathbb{R}$. Prove that the following statements are equivalent:
(i) E is (Lebesgue) measurable.
(ii) For every $\varepsilon > 0$, there exists an open set $O \supset E$ with $\mu_L^*(O \setminus E) \leq \varepsilon$.
(iii) There exists a G_δ-set $G \supset E$ with $\mu_L^*(G \setminus E) = 0$.

Solution
• $(i) \Rightarrow (ii)$ Suppose that E is measurable. Then

$$\forall \varepsilon > 0, \exists \text{ open set } O: \ O \supset E \ \text{ and } \ \mu_L^*(E) \leq \mu_L^*(O) \leq \mu_L^*(E) + \varepsilon. \quad (1)$$

Since E is measurable, with O as a testing set in the Carathéodory condition satisfied by E, we have

$$\mu_L^*(O) = \mu_L^*(O \cap E) + \mu_L^*(O \cap E^c) = \mu_L^*(E) + \mu_L^*(O \setminus E). \quad (2)$$

If $\mu_L^*(E) < \infty$, then from (1) and (2) we get

$$\mu_L^*(O) \leq \mu_L^*(E) + \varepsilon \Longrightarrow \mu_L^*(O) - \mu_L^*(E) = \mu_L^*(O \setminus E) \leq \varepsilon.$$

If $\mu_L^*(E) = \infty$, let $E_n = E \cap (n-1, n]$ for $n \in \mathbb{Z}$. Then $(E_n)_{n\in\mathbb{Z}}$ is a disjoint sequence in $\mathcal{M}_L$ with

$$\bigcup_{n\in\mathbb{Z}} E_n = E \quad \text{and} \quad \mu_L(E_n) \leq \mu_L\big((n-1, n]\big) = 1.$$

Now, for every $\varepsilon > 0$, there is an open set O_n such that

$$O_n \supset E_n \quad \text{and} \quad \mu_L(O_n \setminus E_n) \leq \frac{1}{3}.\frac{\varepsilon}{2^{|n|}}.$$

Let $O = \bigcup_{n\in\mathbb{Z}})O_n$, then O is open and $O \supset E$, and

$$\begin{aligned} O \setminus E &= \left(\bigcup_{n\in\mathbb{Z}} O_n\right) \setminus \left(\bigcup_{n\in\mathbb{Z}} E_n\right) = \left(\bigcup_{n\in\mathbb{Z}} O_n\right) \cap \left(\bigcup_{n\in\mathbb{Z}} E_n\right)^c \\ &= \bigcup_{n\in\mathbb{Z}} \left[O_n \cap \left(\bigcup_{n\in\mathbb{Z}} E_n\right)^c\right] = \bigcup_{n\in\mathbb{Z}} \left[O_n \setminus \left(\bigcup_{n\in\mathbb{Z}} E_n\right)\right] \\ &\subset \bigcup_{n\in\mathbb{Z}} (O_n \setminus E_n). \end{aligned}$$

Then we have

$$\begin{aligned} \mu_L^*(O \setminus E) &\leq \mu_L^*\left(\bigcup_{n\in\mathbb{Z}} (O_n \setminus E_n)\right) \leq \sum_{n\in\mathbb{Z}} \mu_L^*(O_n \setminus E) \\ &\leq \sum_{n\in\mathbb{Z}} \frac{1}{3}.\frac{\varepsilon}{2^{|n|}} = \frac{1}{3}\varepsilon + 2\sum_{n\in\mathbb{N}} \frac{1}{3}.\frac{\varepsilon}{2^n} \\ &= \frac{1}{3}\varepsilon + \frac{2}{3}\varepsilon = \varepsilon. \end{aligned}$$

This shows that (ii) satisfies.

• $(ii) \Rightarrow (iii)$ Assume that E satisfies (ii). Then for $\varepsilon = \frac{1}{n},\ n \in \mathbb{N}$, there is an open set O_n such that

$$O_n \supset E_n \quad \text{and} \quad \mu_L(O_n \setminus E_n) \leq \frac{1}{n},\ \forall n \in \mathbb{N}.$$

Let $G = \bigcap_{n\in\mathbb{N}} O_n$. Then G is a G_δ-set containing E. Now

$$G \subset O \Longrightarrow \mu_L^*(G \setminus E) \leq \mu_L^*(O_n \setminus E) \leq \frac{1}{n},\ \forall n \in \mathbb{N}.$$

Thus $\mu_L^*(G \setminus E) = 0$. This shows that E satisfies (iii).

• $(iii) \Rightarrow (i)$ Assume that E satisfies (iii). Then there exists a G_δ-set G such that

$$G \supset E \text{ and } \mu_L^*(G \setminus E) = 0.$$

Now $\mu_L^*(G \setminus E) = 0$ implies that $G \setminus E$ is (Lebesgue) measurable. Since $E \subset G$, we can write $E = G \setminus (G \setminus E)$. Then the fact that G and $G \setminus E$ are (Lebesgue) measurable implies that E is (Lebesgue) measurable. ■

Problem 17(Similar problem)
Let $E \subset \mathbb{R}$. Prove that the following statements are equivalent:
(i) E is (Lebesgue) measurable.
(ii) For every $\varepsilon > 0$, there exists an closed set $C \subset E$ with $\mu_L^*(E \setminus C) \leq \varepsilon$.
(iii) There exists a F_σ-set $F \subset E$ with $\mu_L^*(E \setminus F) = 0$.

Problem 18
Let $\mathbb{Q}$ be the set of all rational numbers in $\mathbb{R}$. For any $\varepsilon > 0$, construct an open set $O \subset \mathbb{R}$ such that

$$O \supset \mathbb{Q} \text{ and } \mu_L^*(O) \leq \varepsilon.$$

Solution
Since $\mathbb{Q}$ is countable, we can write $\mathbb{Q} = \{r_1, r_2, ...\}$. For any $\varepsilon > 0$, let

$$I_n = \left(r_n - \frac{\varepsilon}{2^{n+1}}, r_n + \frac{\varepsilon}{2^{n+1}}\right), \quad n \in \mathbb{N}.$$

Then I_n is open and $O = \bigcup_{n=1}^\infty I_n$ is also open. We have, for every $n \in \mathbb{N}$, $r_n \in I_n$. Therefore $O \supset \mathbb{Q}$.
Moreover,

$$\begin{aligned} \mu_L^*(O) = \mu_L^*\left(\bigcup_{n=1}^\infty I_n\right) &\leq \sum_{n=1}^\infty \mu_L^*(I_n) \\ &= \sum_{n=1}^\infty \frac{2\varepsilon}{2^{n+1}} \\ &= \varepsilon \sum_{n=1}^\infty \frac{1}{2^n} = \varepsilon. \quad ■ \end{aligned}$$

Problem 19
Let $\mathbb{Q}$ be the set of all rational numbers in $\mathbb{R}$.
(a) Show that $\mathbb{Q}$ is a null set in $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$.
(b) Show that $\mathbb{Q}$ is a F_σ-set.
(c) Show that there exists a G_δ-set G such that $G \supset \mathbb{Q}$ and $\mu_L(G) = 0$.
(d) Show that the set of all irrational numbers in $\mathbb{R}$ is a G_δ-set.

Solution
(a) Since $\mathbb{Q}$ is countable, we can write $\mathbb{Q} = \{r_1, r_2, ...\}$. Each $\{r_n\},\ n \in \mathbb{N}$ is closed, so $\{r_n\} \in \mathcal{B}_{\mathbb{R}}$. Since $\mathcal{B}_{\mathbb{R}}$ is a σ-algebra,

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\} \in \mathcal{B}_{\mathbb{R}}.$$

Since $\mu_L(\{r_n\}) = 0$, we have

$$\mu_L(\mathbb{Q}) = \sum_{n=1}^{\infty} \mu_L(\{r_n\}) = 0.$$

Thus, $\mathbb{Q}$ is a null set in $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$.

(b) Since $\{r_n\}$ is closed and $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$, $\mathbb{Q}$ is a F_σ-set.

(c) By (a), $\mu_L(\mathbb{Q}) = 0$. This implies that, for every $n \in \mathbb{N}$, there exists an open set G_n such that

$$G_n \supset \mathbb{Q} \text{ and } \mu_L(G_n) < \frac{1}{n}.$$

If $G = \bigcap_{n=1}^{\infty} G_n$ then G is a G_δ-set and $G \supset \mathbb{Q}$. Furthermore,

$$\mu_L(G) \leq \mu_L(G_n) < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

This implies that $\mu_L(G) = 0$.

(d) By (b), $\mathbb{Q}$ is a F_σ-set, so $\mathbb{R} \setminus \mathbb{Q}$, the set of all irrational numbers in $\mathbb{R}$, is a G_δ-set. ■

Problem 20
Let $E \in \mathcal{M}_L$ with $\mu_L(E) > 0$. Prove that for every $\alpha \in (0, 1)$, there exists a finite open interval I such that

$$\alpha \mu_L(I) \leq \mu_L(E \cap I) \leq \mu_L(I).$$

Solution

• Consider first the case where $0 < \mu_L(E) < \infty$. For any $\alpha \in (0,1)$, set $\frac{1}{\alpha} = 1 + a$. Since $a > 0,\ 0 < \varepsilon = a\mu_L(E) < \infty$. By the regularity property of μ_L^* (Property 7), there exists an open set $O \supset E$ such that[1]

$$\mu_L(O) \leq \mu_L(E) + a\mu_L(E) = (1+a)\mu_L(E) = \frac{1}{\alpha}\mu_L(E) < \infty. \quad (i)$$

Now since O is an open set in $\mathbb{R}$, it is union of a disjoint sequence (I_n) of open intervals in $\mathbb{R}$:

$$O = \bigcup_{n\in\mathbb{N}} I_n \Longrightarrow \mu_L(O) = \sum_{n\in\mathbb{N}} \mu_L(I_n). \quad (ii)$$

Since $E \subset O$, we have

$$\mu_L(E) = \mu_L(E \cap O) = \mu_L\left(E \cap \bigcup_{n\in\mathbb{N}} I_n\right) = \sum_{n\in\mathbb{N}} \mu_L(E \cap I_n). \quad (iii)$$

From (i), (ii) and (iii) it follows that

$$\sum_{n\in\mathbb{N}} \mu_L(I_n) \leq \frac{1}{\alpha} \sum_{n\in\mathbb{N}} \mu_L(E \cap I_n).$$

Note that all terms in this inequality are positive, so that there exists at least one $n_0 \in \mathbb{N}$ such that

$$\mu_L(I_{n_0}) \leq \frac{1}{\alpha}\mu_L(E \cap I_{n_0}).$$

Since $\mu_L(O)$ is finite, all intervals I_n are finite intervals in $\mathbb{R}$. Let $I := I_{n_0}$, then I is a finite open interval satisfying conditions:

$$\alpha\mu_L(I) \leq \mu_L(E \cap I) \leq \mu_L(I).$$

• Now consider that case $\mu_L(E) = \infty$. By the σ-finiteness of the Lebesgue measure space, there exists a measurable subset E_0 of E such that $0 < \mu_L(E_0) < \infty$. Then using the first part of the solution, we obtain

$$\alpha\mu_L(I) \leq \mu_L(E_0 \cap I) \leq \mu_L(E \cap I) \leq \mu_L(I). \quad \blacksquare$$

[1] Recall that for (Lebesgue) measurable set A, $\mu_L^*(A) = \mu_L(A)$.

Problem 21
Let f be a real-valued function on (a,b) such that f' exists and satisfies

$$|f'(x)| \leq M \quad \textit{for all} \quad x \in (a,b) \quad \textit{and for some} \quad M \geq 0.$$

Show that for every $E \subset (a,b)$ we have

$$\mu_L^*(f(E)) \leq M\mu_L^*(E).$$

Solution
If $M = 0$ then $f'(x) = 0, \ \forall x \in (a,b)$. Hence, $f(x) = y_0, \ \forall x \in (a,b)$. Thus, for any $E \subset (a,b)$ we have

$$\mu_L^*(f(E)) = 0.$$

The inequality holds. Suppose $M > 0$. For all $x, y \in (a,b)$, by the Mean Value Theorem, we have

$$\begin{aligned} |f(x) - f(y)| &= |x-y||f'(c)|, \quad \text{for some} \quad c \in (a,b) \\ &\leq M|x-y|. \quad (*) \end{aligned}$$

By definition of the outer measure, for any $E \subset (a,b)$ we have

$$\mu_L^*(E) = \inf \sum_{n=1}^{\infty}(b_n - a_n),$$

where $\{I_n = (a_n, b_n), \ n \in \mathbb{N}\}$ is a covering class of E. By (*) we have

$$\begin{aligned} \sum_{n=1}^{\infty} |f(b_n) - f(a_n)| &\leq M \sum_{n=1}^{\infty} |b_n - a_n| \\ &\leq M \inf \sum_{n=1}^{\infty} |b_n - a_n| \\ &\leq M\mu_L^*(E). \end{aligned}$$

Infimum takes over all covering classes of E. Thus,

$$\mu_L^*(f(E)) = \inf \sum_{n=1}^{\infty} |f(b_n) - f(a_n)| \leq M\mu_L^*(E). \quad \blacksquare$$

Problem 22
(a) Let $E \subset \mathbb{R}$. *Show that* $\mathcal{F} = \{\varnothing, E, E^c, \mathbb{R}\}$ *is the* σ*-algebra of subsets of* $\mathbb{R}$ *generated by* $\{E\}$
(b) If $\mathcal{S}$ *and* $\mathcal{T}$ *are collections of subsets of* $\mathbb{R}$, *then*

$$\sigma(\mathcal{S} \cup \mathcal{T}) = \sigma(\mathcal{S}) \cup \sigma(\mathcal{T}).$$

Is the statement true? Why?

Solution
(a)It is easy to check that $\mathcal{F}$ is a σ-algebra.
Note first that $\{E\} \subset \mathcal{F}$. Hence

$$\sigma(\{E\}) \subset \mathcal{F}. \quad (i)$$

On the other hand, since $\sigma(\{E\})$ is a σ-algebra, so $\varnothing, \mathbb{R} \in \sigma(\{E\})$. Also, since $E \in \sigma(\{E\})$, so $E^c \in \sigma(\{E\})$. Hence

$$\mathcal{F} \subset \sigma(\{E\}). \quad (ii)$$

From (i) and (ii) it follows that

$$\mathcal{F} = \sigma(\{E\}).$$

(b) No. Here is why.
Take $\mathcal{S} = \{(,1]\}$ and $\mathcal{T} = \{(1,2]\}$. Then, by part (a),

$$\sigma(\mathcal{S}) = \{\varnothing, (0,1], (0,1]^c, \mathbb{R}\} \text{ and } \sigma(\mathcal{T}) = \{\varnothing, (1,2], (1,2]^c, \mathbb{R}\}.$$

Therefore

$$\sigma(\mathcal{S}) \cup \sigma(\mathcal{T}) = \{\varnothing, (0,1], (0,1]^c, (1,2], (1,2]^c, \mathbb{R}\}.$$

We have

$$(0,1] \cup (1,2] = (0,2] \notin \sigma(\mathcal{S}) \cup \sigma(\mathcal{T}).$$

Hence $\sigma(\mathcal{S}) \cup \sigma(\mathcal{T})$ is not a σ-algebra. But, by definition, $\sigma(\mathcal{S} \cup \mathcal{T})$ is a σ-algebra. And hence it cannot be equal to $\sigma(\mathcal{S}) \cup \sigma(\mathcal{T})$. ■

Problem 23
Consider $\mathcal{F} = \{E \in \mathbb{R} : \text{ either } E \text{ is countable or } E^c \text{ is countable}\}$.
(a) Show that $\mathcal{F}$ *is a* σ*-algebra and* $\mathcal{F}$ *is a proper sub-*σ*-algebra of the* σ*-algebra* $\mathcal{B}_{\mathbb{R}}$.
(b) Show that $\mathcal{F}$ *is the* σ*-algebra generated by* $\big\{\{x\} : x \in \mathbb{R}\big\}$.
(c) Find a measure $\lambda : \mathcal{F} \to [0, \infty]$ *such that the only* λ*-null set is* $\varnothing$.

Solution

(a) We check conditions of a σ-algebra:

• It is clear that $\varnothing$ is countable, so $\varnothing \in \mathcal{F}$.

• Suppose $E \in \mathcal{F}$. Then $E \subset \mathbb{R}$ and E is countable or E^c is countable. This is equivalent to $E^c \subset \mathbb{R}$ and E^c is countable or E is countable. Thus,

$$E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}.$$

• Suppose $E_1, E_2, ... \in \mathcal{F}$. Either all E_n's are countable, so $\bigcup_{n=1}^{\infty} E_n$ is countable. Hence $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$. Or there exists some $E_{n_0} \in \mathcal{F}$ which is not countable. By definition, $E_{n_0}^c$ must be countable. Now

$$\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c \subset E_{n_0}.$$

This implies that $(\bigcup_{n=1}^{\infty} E_n)^c$ is countable. Thus

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}.$$

Finally, $\mathcal{F}$ is a σ-algebra. $\square$

Recall that $\mathcal{B}_{\mathbb{R}}$ is the σ-algebra generated by the family of open sets in $\mathbb{R}$. It is also generated by the family of closed sets in $\mathbb{R}$. Now suppose $E \in \mathcal{F}$. If E is countable then we can write

$$E = \{x_1, x_2, ...\} = \bigcup_{n=1}^{\infty} \{x_n\}.$$

Each $\{x_n\}$ is a closed set in $\mathbb{R}$, so belongs to $\mathcal{B}_{\mathbb{R}}$. Hence $E \in \mathcal{B}_{\mathbb{R}}$. Therefore,

$$\mathcal{F} \subset \mathcal{B}_{\mathbb{R}}.$$

$\mathcal{F}$ is a proper subset of $\mathcal{B}_{\mathbb{R}}$. Indeed, $[0, 1] \in \mathcal{B}_{\mathbb{R}}$ and $[0, 1] \notin \mathcal{F}$. $\square$

(b) Let $\mathcal{S} = \big\{\{x\} : \ x \in \mathbb{R}\big\}$. Clearly, $\mathcal{S} \subset \mathcal{F}$, and so

$$\sigma(\mathcal{S}) \subset \mathcal{F}.$$

Now take $E \in \mathcal{F}$ and $E \neq \varnothing$. If E is countable then we can write

$$E = \bigcup_{n=1}^{\infty} \underbrace{\{x_n\}}_{\in \mathcal{S}} \in \sigma(\mathcal{S}).$$

Hence

$$\mathcal{F} \subset \sigma(\mathcal{S}).$$

Thus

$$\sigma(\mathcal{S}) = \mathcal{F}.$$

(c) Define the set function $\lambda : \mathcal{F} \to [0, \infty]$ by

$$\lambda(E) = \begin{cases} |E| & \text{if } E \text{ is finite} \\ \infty & \text{otherwise.} \end{cases}$$

We can check that λ is a measure. If $E \neq \varnothing$ then $\lambda(E) > 0$ for every $E \in \mathcal{F}$. ■

Problem 24
For $E \in \mathfrak{M}_L$ with $\mu_L(E) < \infty$, define a real-valued function φ_E on $\mathbb{R}$ by setting

$$\varphi_E(x) := \mu_L(E \cap (-\infty, x]) \quad \text{for} \quad x \in \mathbb{R}.$$

(a) Show that φ_E is an increasing function on $\mathbb{R}$.
(b) Show that φ_E satisfies the Lipschitz condition on $\mathbb{R}$, that is,

$$|\varphi_E(x') - \varphi_E(x'')| \leq |x' - x''| \quad \text{for} \quad x', x'' \in \mathbb{R}.$$

Solution
(a) Let $x, y \in \mathbb{R}$. Suppose $x < y$. It is clear that $(-\infty, x] \subset (-\infty, y]$. Hence, $E \cap (-\infty, x] \subset E \cap (-\infty, y]$ for $E \in \mathfrak{M}_L$. By the monoticity of μ_L we have

$$\varphi_E(x) = \mu_L(E \cap (-\infty, x]) \leq \mu_L(E \cap (-\infty, y]) = \varphi_E(y).$$

Thus φ_E is increasing on $\mathbb{R}$.

(b) Suppose $x' < x''$ we have

$$E \cap (x', x''] = (E \cap (-\infty, x'']) \setminus (E \cap (-\infty, x']).$$

It follows that

$$\begin{aligned} \varphi_E(x'') - \varphi_E(x') &= \mu_L(E \cap (-\infty, x'']) - \mu_L(E \cap (-\infty, x']) \\ &= \mu_L(E \cap (x', x'']) \\ &\leq \mu_L\big((x', x'']\big) = x'' - x'. \quad ■ \end{aligned}$$

Problem 25
Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $\mu_L(E) = 1$. Show that there exists a Lebesgue measurable set $A \subset E$ such that $\mu_L(A) = \frac{1}{2}$.

Solution
Define the function $f : \mathbb{R} \to [0, 1]$ by

$$f(x) = \mu_L\big(E \cap (-\infty, x]\big),\ x \in \mathbb{R}.$$

By Problem 23, we have

$$|f(x) - f(y)| \leq |x - y|,\ \forall x, y \in \mathbb{R}.$$

Hence f is (uniformly) continuous on $\mathbb{R}$. Since

$$\lim_{x\to-\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 1,$$

by the Mean Value Theorem, we have

$$\exists x_0 \in \mathbb{R} \text{ such that } f(x_0) = \frac{1}{2}.$$

Set $A = E \cap (-\infty, x_0]$. Then we have

$$A \subset E \quad \text{and} \quad \mu_L(A) = \frac{1}{2}. \qquad \blacksquare$$

Chapter 3

Measurable Functions

Remark:
From now on, *measurable* means Lebesgue measurable. Also *measure* means Lebesgue measure, and we write μ instead of μ_L for Lebesgue measure.

1. Definition, basic properties

Proposition 6 *(Equivalent conditions)*
Let f be an extended real-valued function whose domain D is measurable. Then the following statements are equivalent:

1. *For each real number a, the set $\{x \in D : f(x) > a\}$ is measurable.*
2. *For each real number a, the set $\{x \in D : f(x) \geq a\}$ is measurable.*
3. *For each real number a, the set $\{x \in D : f(x) < a\}$ is measurable.*
4. *For each real number a, the set $\{x \in D : f(x) \leq a\}$ is measurable.*

Definition 11 *(Measurable function)*
An extended real-valued function f is said to be measurable if its domain is measurable and if it satisfies one of the four statements of Proposition 6.

Proposition 7 *(Operations)*
Let f, g be two measurable real-valued functions defined on the same domain and c a constant. Then the functions $f + c, cf, f + g$, and fg are also measurable.

Note:
A function f is said to be *Borel measurable* if for each $\alpha \in \mathbb{R}$ the set $\{x : f(x) > \alpha\}$ is a Borel set. Every Borel measurable function is Lebesgue measurable.

2. Equality almost everywhere
• A property is said to hold *almost everywhere* (abbreviated a.e.) if the set of points where it fails to hold is a set of measure zero.
• We say that $f = g$ a.e. if f and g have the same domain and $\mu\big(\{x \in D : f(x) \neq g(x)\}\big) = 0$. Also we say that the sequence (f_n) converges to f a.e. if the set $\{x : f_n(x) \nrightarrow f(x)\}$ is a null set.

Proposition 8 *(Measurable functions)*
If a function f is measurable and $f = g$ a.e., then g is measurable.

3. Sequence of measurable functions

Proposition 9 *(Monotone sequence)*
Let (f_n) be a monotone sequence of extended real-valued measurable functions on the same measurable domain D. Then $\lim_{n\to\infty} f_n$ exists on D and is measurable.

Proposition 10 *Let (f_n) be a sequence of extended real-valued measurable functions on the same measurable domain D. Then $\max\{f_1, ..., f_n\}$, $\min\{f_1, ..., f_n\}$, $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$, $\sup_{n\in\mathbb{N}}$, $\inf_{n\in\mathbb{N}}$ are all measurable.*

Proposition 11 *If f is continuous a.e. on a measurable set D, then f is measurable.*

* * **

Problem 26
Let D be a dense set in $\mathbb{R}$. Let f be an extended real-valued function on $\mathbb{R}$ such that $\{x: f(x) > \alpha\}$ is measurable for each $\alpha \in D$. Show that f is measurable.

Solution
Let β be an arbitrary real number. For each $n \in \mathbb{N}$, there exists $\alpha_n \in D$ such that $\beta < \alpha_n < \beta + \frac{1}{n}$ by the density of D. Now

$$\{x: f(x) > \beta\} = \bigcup_{n=1}^{\infty} \left\{x: f(x) \geq \beta + \frac{1}{n}\right\} = \bigcup_{n=1}^{\infty} \{x: f(x) > \alpha_n\}.$$

Since $\bigcup_{n=1}^{\infty} \{x: f(x) > \alpha_n\}$ is measurable (as countable union of measurable sets), $\{x: f(x) > \beta\}$ is measurable. Thus, f is measurable. ■

Problem 27
Let f be an extended real-valued measurable function on $\mathbb{R}$. Prove that $\{x: f(x) = \alpha\}$ is measurable for any $\alpha \in \overline{\mathbb{R}}$.

Solution

- For $\alpha \in \mathbb{R}$, we have

$$\{x: \ f(x) = \alpha\} = \underbrace{\{x: \ f(x) \le \alpha\}}_{\text{measurable}} \setminus \underbrace{\{x: \ f(x) < \alpha\}}_{\text{measurable}}.$$

Thus $\{x: \ f(x) = \alpha\}$ is measurable.

- For $\alpha = \infty$, we have

$$\{x: \ f(x) = \infty\} = \mathbb{R} \setminus \{x: \ f(x) < \infty\} = \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} \underbrace{\{x: \ f(x) \le n\}}_{\text{measurable}}.$$

Thus $\{x: \ f(x) = \infty\}$ is measurable.

- For $\alpha = -\infty$, we have

$$\{x: \ f(x) = -\infty\} = \mathbb{R} \setminus \{x: \ f(x) > -\infty\} = \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} \underbrace{\{x: \ f(x) \ge -n\}}_{\text{measurable}}.$$

Thus $\{x: \ f(x) = \infty\}$ is measurable. ■

Problem 28

(a). Let D and E be measurable sets and f a function with domain $D \cup E$. Show that f is measurable if and only if its restriction to D and E are measurable.
(b). Let f be a function with measurable domain D. Show that f is measurable if and only if the function g defined by

$$g(x) = \begin{cases} f(x) & \text{for } x \in D \\ 0 & \text{for } x \notin D \end{cases}$$

is measurable.

Solution

(a) Suppose that f is measurable. Since D and E are measurable subsets of $D \cup E$, the restrictions $f|_D$ and $f|_E$ are measurable.
Conversely, suppose $f|_D$ and $f|_E$ are measurable. For any $\alpha \in \mathbb{R}$, we have

$$\{x: \ f(x) > \alpha\} = \{x \in D: \ f|_D(x) > \alpha\} \cup \{x \in E: \ f|_E(x) > \alpha\}.$$

Each set on the right hand side is measurable, so $\{x: \ f(x) > \alpha\}$ is measurable. Thus, f is measurable.

(b) Suppose that f is measurable. If $\alpha \geq 0$, then $\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\}$, which is measurable. If $\alpha < 0$, then $\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup D^c$, which is measurable. Hence, g is measurable.
Conversely, suppose that g is measurable. Since $f = g|_D$ and D is measurable, f is measurable. ■

Problem 29
Let f be measurable and B a Borel set. Then $f^{-1}(B)$ is a measurable set.

Solution
Let $\mathcal{C}$ be the collection of all sets E such that $f^{-1}(E)$ is measurable. We show that $\mathcal{C}$ is a σ-algebra. Suppose $E \in \mathcal{C}$. Since

$$f^{-1}(E^c) = \left(f^{-1}(E)\right)^c,$$

which is measurable, so $E^c \in \mathcal{C}$. Suppose (E_n) is a sequence of sets in $\mathcal{C}$. Since

$$f^{-1}\left(\bigcup_n E_n\right) = \bigcup_n f^{-1}(E_n),$$

which is measurable, so $\bigcup_n E_n \in \mathcal{C}$. Thus, $\mathcal{C}$ is a σ-algebra.
Next, we show that all intervals (a, b), for any extended real numbers a, b with $a < b$, belong to $\mathcal{C}$. Since f is measurable, $\{x : f(x) > a\}$ and $\{x : f(x) < b\}$ are measurable. It follows that (a, ∞) and $(-\infty, b) \in \mathcal{C}$. Furtheremore, we have

$$(a, b) = (-\infty, b) \cap (a, \infty),$$

so $(a, b) \in \mathcal{C}$. Thus, $\mathcal{C}$ is a σ-algebra containing all open intervals, so it contains all Borel sets. Hence $f^{-1}(B)$ is measurable. ■

Problem 30
Show that if f is measurable real-valued function and g a continuous function defined on $\mathbb{R}$, then $g \circ f$ is measurable.

Solution
For any $\alpha \in \mathbb{R}$,

$$\{x : (g \circ f)(x) > \alpha\} = (g \circ f)^{-1}((\alpha, \infty)) = f^{-1}\Big(g^{-1}\big((\alpha, \infty)\big)\Big).$$

By the continuity of g, $g^{-1}\big((\alpha,\infty)\big)$ is an open set, so a Borel set. By Problem 24, the last set is measurable. Thus, $g\circ f$ is measurable. $\square$

Problem 31
Let f be an extended real-valued function defined on a measurable set $D\subset\mathbb{R}$.
(a) Show that if $\{x\in D:\ f(x)<r\}$ is measurable in $\mathbb{R}$ for every $r\in\mathbb{Q}$, then f is measurable on D.
(b) What subsets of $\mathbb{R}$ other than $\mathbb{Q}$ have this property?
(c) Show that if f is measurable on D, then there exists a countable sub-collection $\mathcal{C}\subset\mathcal{M}_L$, depending on f, such that f is $\sigma(\mathcal{C})$-measurable on D.
(Note: $\sigma(\mathcal{C})$ is the σ-algebra generated by $\mathcal{C}$.)

Solution
(a) To show that f is measurable on D, we show that $\{x\in D:\ f(x)<a\}$ is measurable for every $a\in\mathbb{R}$. Let $I=\{r\in\mathbb{Q}:\ r<a\}$. Then I is countable , and we have

$$\{x\in D:\ f(x)<a\}=\bigcup_{r\in I}\{x\in D:\ f(x)<r\}.$$

Since $\{x\in D:\ f(x)<r\}$ is measurable, $\bigcup_{r\in I}\{x\in D:\ f(x)<r\}$ is measurable. Thus, $\{x\in D:\ f(x)<a\}$ is measurable.

(b) Here is the answer to the question:

Claim 1 : *If $E\subset\mathbb{R}$ is dense in $\mathbb{R}$, then E has the property in (a), that is, if $\{x\in D:\ f(x)<r\}$ is measurable for every $r\in E$ then f is measurable on D.*
Proof.
Given any $a\in\mathbb{R}$, the interval $(a-1,a)$ intersects E since E is dense. Pick some $r_1\in E\cap(a-1,a)$. Now the interval (r_1,a) intersects E for the same reason. Pick some $r_2\in E\cap(r_1,a)$. Repeating this process, we obtain an increasing sequence (r_n) in E which converges to a.
By assumption, $\{x\in D:\ f(x)<r_n\}$ is measurable, so we have

$$\{x\in D:\ f(x)<a\}=\bigcup_{n\in\mathbb{N}}\{x\in D:\ f(x)<r_n\}\ \text{ is measurable .}$$

Thus, f is measurable on D.

Claim 2 : *If $E\subset\mathbb{R}$ is not dense in $\mathbb{R}$, then E does not have the property in (a).*
Proof.
Since E is not dense in $\mathbb{R}$, there exists an interval $[a,b]\subset E$. Let F be a non

measurable set in $\mathbb{R}$. We define a function f as follows:

$$f(x) = \begin{cases} a & \text{if } x \in F^c \\ b & \text{if } x \in F. \end{cases}$$

For $r \in E$, by definition of F, we observe that

- If $r < a$ then $f^{-1}([-\infty, r)) = \emptyset$.
- If $r > b$ then $f^{-1}([-\infty, r)) = \overline{\mathbb{R}}$.
- If $r = \frac{a+b}{2}$ then $f^{-1}([-\infty, r)) = F^c$.

Since F is non measurable, F^c is also non measurable. Through the above observation, we see that

$$\left\{x \in D : \ f(x) < \frac{a+b}{2}\right\} \ \text{ non measurable.}$$

Thus, f is not measurable.

Conclusion : Only subsets of $\mathbb{R}$ which are dense in $\mathbb{R}$ have the property in (a).

(c) Let $\mathcal{C} = \{C_r\}_{r\in\mathbb{Q}}$ where $C_r = \{x \in D : \ f(x) < r\}$ for every $r \in \mathbb{Q}$. Clearly, $\mathcal{C}$ is a countable family of subsets of $\mathbb{R}$. Since f is measurable, C_r is measurable. Hence, $\mathcal{C} \subset \mathcal{M}_L$. Since $\mathcal{M}_L$ is a σ-algebra, by definition, we must have $\sigma(\mathcal{C}) \subset \mathcal{M}_L$. Let $a \in \mathbb{R}$. Then

$$\{x \in D : \ f(x) < a\} = \bigcup_{r<a} \{x \in D : \ f(x) < r\} = \bigcup_{r<a} C_r.$$

It follows that $\{x \in D : \ f(x) < a\} \in \sigma(\mathcal{C})$.

Thus, f is $\sigma(\mathcal{C})$-measurable on D. ■

Problem 32

Show that the following functions defined on $\mathbb{R}$ are all Borel measurable, and hence Lebesgue measurable also on $\mathbb{R}$:

$$(a) \quad f(x) = \begin{cases} 0 & \textit{if } x \textit{ is rational} \\ 1 & \textit{if } x \textit{ is irrational.} \end{cases} \qquad (b) \quad g(x) = \begin{cases} x & \textit{if } x \textit{ is rational} \\ -x & \textit{if } x \textit{ is irrational.} \end{cases}$$

$$(c) \quad h(x) = \begin{cases} \sin x & \textit{if } x \textit{ is rational} \\ \cos x & \textit{if } x \textit{ is irrational.} \end{cases}$$

Solution
(a) For any $a \in \mathbb{R}$, let $E = \{x \in D : \; f(x) < a\}$.

- If $a > 1$ then $E = \mathbb{R}$, so $E \in \mathcal{B}_{\mathbb{R}}$ (Borel measurable).
- If $0 < a \leq 1$ then $E = \mathbb{Q}$, so $E \in \mathcal{B}_{\mathbb{R}}$ (Borel measurable).
- If $a \leq 0$ then $E = \varnothing$, so $E \in \mathcal{B}_{\mathbb{R}}$ (Borel measurable).

Thus, f is Borel measurable.

(b) Consider g_1 defined on $\mathbb{Q}$ by $g_1(x) = x$, then $g|_{\mathbb{Q}} = g_1$. Consider g_2 defined on $\mathbb{R} \setminus \mathbb{Q}$ by $g(x) = -x$, then $g|_{\mathbb{R}\setminus\mathbb{Q}} = g_2$. Notice that $\mathbb{R},\ \mathbb{R}\setminus\mathbb{Q} \in \mathcal{B}_{\mathbb{R}}$ (Borel measurable). For any $a \in \mathbb{R}$, we have

$$\{x \in D : \; f_1(x) < a\} = [-\infty, a) \cap \mathbb{Q} \in \mathcal{B}_{\mathbb{R}} \quad \text{(Borel measurable)},$$

and

$$\{x \in D : \; f_2(x) < a\} = [-\infty, a) \cap (\mathbb{R} \setminus \mathbb{Q}) \in \mathcal{B}_{\mathbb{R}} \quad \text{(Borel measurable)}.$$

Thus, g is Borel measurable.

(c) Use the same way as in (b). ■

Problem 33
Let f be a real-valued increasing function on $\mathbb{R}$. Show that f is Borel measurable, and hence Lebesgue measurable also on $\mathbb{R}$.

Solution
For any $a \in \mathbb{R}$, let $E = \{x \in D : \; f(x) \geq a\}$. Let $\alpha = \inf E$. Since f is increasing,

- if $\operatorname{Im}(f) \subset (-\infty, a)$ then $E = \varnothing$.
- if $\operatorname{Im}(f) \not\subseteq (-\infty, a)$ then E is either (α, ∞) or $[\alpha, \infty)$.

Since $\varnothing,\ (\alpha, \infty),\ [\alpha, \infty)$ are Borel sets, so f is Borel measurable. ■

Problem 34
If (f_n) is a sequence of measurable functions on $D \subset \mathbb{R}$, then show that

$$\{x \in D : \; \lim_{n\to\infty} f_n(x) \;\; \textit{exists}\} \;\; \textit{is measurable.}$$

Solution

Recall that if f_n's are measurable, then $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$ and $g(x) = \limsup_{n\to\infty} f_n - \liminf_{n\to\infty} f_n$ are also measurable, and if h is measurable then $\{x \in D : \ h(x) = \alpha\}$ is measurable (Problem 22).
Now we have

$$E = \{x \in D : \lim_{n\to\infty} f_n(x) \ \text{ exists}\} = \{x \in D : \ g(x) = 0\}.$$

Thus, E is measurable. ■

Problem 35
(a) If $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to \mathbb{R}$ is measurable then $g \circ f$ is measurable.
(b) If f is measurable then $|f|$ is measurable. Does the converse hold?

Solution

(a) For any $a \in \mathbb{R}$, then

$$\begin{aligned} E = \{x : \ (g \circ f)(x) < a\} &= (g \circ f)^{-1}(-\infty, a) \\ &= f^{-1}\left(g^{-1}(-\infty, a)\right). \end{aligned}$$

Since g is continuous, $g^{-1}(-\infty, a)$ is open. Then there is a family of open disjoint intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $g^{-1}(-\infty, a) = \bigcup_{n\in\mathbb{N}} I_n$. Hence,

$$E = f^{-1}\left(\bigcup_{n\in\mathbb{N}} I_n\right) = \bigcup_{n\in\mathbb{N}} f^{-1}(I_n).$$

Since f is measurable, $f^{-1}(I_n)$ is measurable. Hence E is measurable. This tells us that $g \circ f$ is measurable.

(b) If $g(u) = |u|$ then g is continuous. We have

$$(g \circ f)(x) = g(f(x)) = |f(x)|.$$

By part (a), $g \circ f = |f|$ is measurable.
The converse is not true.
Let E be a non-measurable subset of $\mathbb{R}$. Consider the function:

$$f(x) = \begin{cases} 1 & \text{if } x \in E \\ -1 & \text{if } x \notin E. \end{cases}$$

Then $f^{-1}(\frac{1}{2}, \infty) = E$, which is not measurable. Since $(\frac{1}{2}, \infty)$ is open, so f is not measurable, while $|f| = 1$ is measurable. ■

Problem 36
Let $(f_n:\ n\in\mathbb{N})$ *and* f *be an extended real-valued measurable functions on a measurable set* $D\subset\mathbb{R}$ *such that* $\lim_{n\to\infty} f_n = f$ *on* D. *Then for every* $\alpha\in\mathbb{R}$ *prove that:*

$$(i)\quad \mu\{x\in D:\ f(x)>\alpha\}\le \liminf_{n\to\infty}\mu\{x\in D:\ f_n(x)\ge\alpha\}$$
$$(ii)\quad \mu\{x\in D:\ f(x)<\alpha\}\le \liminf_{n\to\infty}\mu\{x\in D:\ f_n(x)\le\alpha\}.$$

Solution
Recall that, for any sequence $(E_n)_{n\in\mathbb{N}}$ of measurable sets,

$$\mu(\liminf_{n\to\infty} E_n)\le \liminf_{n\to\infty}\mu(E_n),\qquad (*)$$
$$\liminf_{n\to\infty} E_n=\bigcup_{n\in\mathbb{N}}\bigcap_{k\ge n}E_k=\lim_{n\to\infty}\bigcap_{k\ge n}E_k.$$

Now for every $\alpha\in\mathbb{R}$, let $E_k=\{x\in D:\ f_k(x)\ge\alpha\}$ for each $k\in\mathbb{N}$. Then

$$\begin{aligned}\liminf_{n\to\infty} E_n &= \lim_{n\to\infty}\bigcap_{k\ge n}E_k\\ &= \lim_{n\to\infty}\bigcap_{k\ge n}\{x\in D:\ f_k(x)\ge\alpha\}\\ &= \{x\in D:\ f(x)>\alpha\}\ \text{ since }\ f_k(x)\to f(x)\ \text{ on }\ D.\end{aligned}$$

Using (*) we get

$$\mu\{x\in D:\ f(x)>\alpha\}\le \liminf_{n\to\infty}\mu\{x\in D:\ f_n\ge\alpha\}.$$

For the second inequality, we use the similar argument.
Let $F_k=\{x\in D:\ f_k(x)\le\alpha\}$ for each $k\in\mathbb{N}$. Then

$$\begin{aligned}\liminf_{n\to\infty} E_n &= \lim_{n\to\infty}\bigcap_{k\ge n}F_k\\ &= \lim_{n\to\infty}\bigcap_{k\ge n}\{x\in D:\ f_k(x)\le\alpha\}\\ &= \{x\in D:\ f(x)<\alpha\}\ \text{ since }\ f_k(x)\to f(x)\ \text{ on }\ D.\end{aligned}$$

Using (*) we get

$$\mu\{x\in D:\ f(x)<\alpha\}\le \liminf_{n\to\infty}\mu\{x\in D:\ f_n\le\alpha\}.\qquad\blacksquare$$

Simple functions

Definition 12 *(Simple function)*
A function $\varphi : X \to \mathbb{R}$ is simple if it takes only a finite number of different values.

Definition 13 *(Canonical representation)*
Let φ be a simple function on X. Let $\{a_1, ..., a_n\}$ the set of distinct valued assumed by φ on D. Let $D_i = \{x \in X : \ \varphi(x) = a_i\}$ for $i = 1, ..., n$. Then the expression

$$\varphi = \sum_{i=1}^{n} a_i \chi_{D_i}$$

is called the canonical representation of φ.

It is evident that $D_i \cap D_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^{n} D_i = X$.

* * **

Problem 37
(a). Show that

$$\begin{aligned} \chi_{A\cap B} &= \chi_A \cdot \chi_B \\ \chi_{A\cup B} &= \chi_A + \chi_B - \chi_A \cdot \chi_B \\ \chi_{A^c} &= 1 - \chi_A. \end{aligned}$$

(b). Show that the sum and product of two simple functions are simple functions.

Solution
(a). We have

$$\begin{aligned} \chi_{A\cap B}(x) = 1 \quad &\Longleftrightarrow \quad x \in A \ \text{ and } \ x \in B \\ &\Longleftrightarrow \quad \chi_A(x) = 1 = \chi_B(x). \end{aligned}$$

Thus,

$$\chi_{A\cap B} = \chi_A \cdot \chi_B.$$

We have

$$\chi_{A\cup B}(x) = 1 \Longleftrightarrow x \in A \cup B.$$

If $x \in A \cap B$ then $\chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x) = 1 + 1 - 1 = 1$.
If $x \notin A \cap B$, then $x \in A \setminus B$ or $x \in B \setminus A$. Then $\chi_A(x) + \chi_B(x) = 1$ and $\chi_A \cdot \chi_B \chi_A(x) + \chi_B(x) = 0$.
Also,

$$\chi_{A \cup B}(x) = 0 \Longleftrightarrow x \notin A \cup B.$$

Then

$$\chi_A(x) = \chi_B(x) = \chi_A(x) \cdot \chi_B(x) = 0.$$

Thus,

$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B.$$

If $\chi_{A^c}(x) = 1$, then $x \notin A$, so $\chi_A(x) = 0$.
If $\chi_{A^c}(x) = 0$, then $x \in A$, so $\chi_A(x) = 1$. Thus,

$$\chi_{A^c} = 1 - \chi_A. \qquad \square$$

(b). Let φ be a simple function having values $a_1, ..., a_n$. Then

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i} \quad \text{where} \quad A_i = \{x : \ \varphi(x) = a_i\}.$$

Similarly, if ψ is a simple function having values $b_1, ..., b_m$. Then

$$\psi = \sum_{j=1}^{m} b_j \chi_{B_j} \quad \text{where} \quad B_j = \{x : \ \psi(x) = b_j\}.$$

Define $C_{ij} := A_i \cap B_j$. Then

$$A_i \subset X = \bigcup_{j=1}^{m} B_j \quad \text{and so} \quad A_i = A_i \cap \bigcup_{j=1}^{m} B_j = \bigcup_{j=1}^{m} C_{ij}.$$

Similarly, we have

$$B_j = \bigcup_{i=1}^{n} C_{ij}.$$

Since the C_{ij}'s are disjoint, this means that (see part (a))

$$\chi_{A_i} = \sum_{j=1}^{m} \chi_{C_{ij}} \quad \text{and} \quad \chi_{B_j} = \sum_{i=1}^{n} \chi_{C_{ij}}.$$

Thus

$$\varphi = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \chi_{C_{ij}} \quad \text{and} \quad \psi = \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \chi_{C_{ij}}.$$

Hence

$$\varphi + \psi = \sum_{i=1}^{n}\sum_{j=1}^{m}(a_i + b_j)\chi_{C_{ij}} \text{ and } \varphi\psi = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \chi_{C_{ij}}.$$

They are simple function. ■

Problem 38
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a simple function defined by

$$\sum_{i=1}^{n} a_i \chi_{A_i} \quad \textit{where} \quad A_i = \{x \in \mathbb{R} : \ \varphi(x) = a_i\}.$$

Prove that φ is measurable if and only if all the A_i's are measurable.

Solution
Assume that A_i is measurable for all $i = 1, ..., n$. Then for any $c \in \mathbb{R}$, we have

$$\{x : \ \varphi(x) > c\} = \bigcup_{a_i > c} A_i.$$

Since every A_i is measurable, $\bigcup_{a_i>c} A_i$ is measurable. Thus $\{x : \ \varphi(x) > c\}$ is measurable. By definition, φ is measurable.
Conversely, suppose φ is measurable. We can suppose $a_1 < a_2 < ... < a_n$. Given $j \in \{1, 2, ..., n\}$, choose c_1 and c_2 such that $a_{j-1} < c_1 < a_j < c_2 < a_{j+1}$. (If $j = 1$ or $j = n$, part of this requirement is empty.) Then

$$\begin{aligned} A_j &= \left(\bigcup_{a_i > c_1} A_i\right) \setminus \left(\bigcup_{a_i > c_2} A_i\right) \\ &= \underbrace{\{x : \ \varphi(x) > c_1\}}_{\text{measurable}} \setminus \underbrace{\{x : \ \varphi(x) > c_2\}}_{\text{measurable}}. \end{aligned}$$

Thus, A_j is measurable for all $j \in \{1, 2, ..., n\}$. ■

Chapter 4

Convergence a.e. and Convergence in Measure

1. Convergence almost everywhere

Definition 14 *Let (f_n) be a sequence extended real-valued measurable functions on a measurable set $D \subset \mathbb{R}$.*
1. We say that $\lim_{n\to\infty} f_n$ exists a.e. on D if there exists a null set N such that $N \subset D$ and $\lim_{n\to\infty} f_n(x)$ exists for every $x \in D \setminus N$.
2. We say that (f_n) converges a.e. on D if $\lim_{n\to\infty} f_n(x)$ exists and $\lim_{n\to\infty} f_n(x) \in \mathbb{R}$ for every $x \in D \setminus N$.

Proposition 12 *(Uniqueness)*
Let (f_n) be a sequence extended real-valued measurable functions on a measurable set $D \subset \mathbb{R}$. Let g_1 and g_2 be two extended real-valued measurable functions on D. Then

$$\Big[\lim_{n\to\infty} f_n = g_1 \text{ a.e. on } D \text{ and } \lim_{n\to\infty} f_n = g_2 \text{ a.e. on } D\Big] \Longrightarrow g_1 = g_2 \text{ a.e. on } D.$$

Theorem 1 *(Borel-Cantelli Lemma)*
For any sequence (A_n) of measurable subsets in $\mathbb{R}$, we have

$$\sum_{n\in\mathbb{N}} \mu(A_n) < \infty \Longrightarrow \mu\big(\limsup_{n\to\infty} A_n\big) = 0.$$

Definition 15 *(Almost uniform convergence)*
Let (f_n) be a sequence extended real-valued measurable functions on a measurable set $D \subset \mathbb{R}$ and f a real-valued measurable functions on D. We say that (f_n) converges a.u. on D to f if for every $\eta > 0$ there exists a measurable set $E \subset D$ such that $\mu(E) < \eta$ and (f_n) converges uniformly to f on $D \setminus E$.

Theorem 2 *(Egoroff)*
Let D be a measurable set with $\mu(D) < \infty$. Let (f_n) be a sequence extended real-valued measurable functions on D and f a real-valued measurable functions on D. If (f_n) converges to f a.e. on D, then (f_n) converges to f a.u. on D.

2. Convergence in measure

Definition 16 *Let (f_n) be a sequence extended real-valued measurable functions on a measurable set $D \subset \mathbb{R}$. We say that (f_n) converges in measure μ on D if there exists a real-valued measurable function f on D such that for every $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mu\{D: \ |f_n - f| \geq \varepsilon\} := \lim_{n\to\infty} \mu\{x \in D: \ |f_n(x) - f(x)| \geq \varepsilon\} = 0.$$

That is,

$$\forall \varepsilon > 0, \ \forall \eta > 0, \ \exists N(\varepsilon, \eta) \in \mathbb{N}: \ \mu\{D: \ |f_n - f| \geq \varepsilon\} < \eta \ \text{ for } \ n \geq N(\varepsilon, \eta).$$

We write $f_n \xrightarrow{\mu} f$ on D for this convergence.

Proposition 13 *(Uniqueness)*
Let (f_n) be a sequence extended real-valued measurable functions on a measurable set $D \subset \mathbb{R}$. Let f and g be two real-valued measurable functions on D. Then

$$[f_n \xrightarrow{\mu} f \text{ on } D \text{ and } f_n \xrightarrow{\mu} g \text{ on } D] \Longrightarrow f = g \text{ a.e. on } D.$$

Proposition 14 *(Equivalent conditions)*

$$(1) \quad [f_n \xrightarrow{\mu} f \text{ on } D] \Longleftrightarrow \forall \varepsilon > 0, \ \exists N(\varepsilon) \in \mathbb{N}: \ \mu\{D: \ |f_n - f| \geq \varepsilon\} < \varepsilon \text{ for } n \geq N(\varepsilon).$$

$$(2) \quad [f_n \xrightarrow{\mu} f \text{ on } D] \Longleftrightarrow \forall m \in \mathbb{N}, \ \exists N(m): \ \mu\Big\{D: \ |f_n - f| \geq \frac{1}{m}\Big\} < \frac{1}{m} \text{ for } m \geq N(m).$$

3. Convergence a.e. and convergence in measure

Theorem 3 *(Lebesgue)*
Let (f_n) be a sequence extended real-valued measurable functions on a measurable set $D \subset \mathbb{R}$. Let f be a real-valued measurable functions on D. Suppose
1. $f_n \to f$ a.e. on D,
2. $\mu(D) < \infty$.
Then $f_n \xrightarrow{\mu} f$ on D.

Theorem 4 *(Riesz)*
Let (f_n) be a sequence extended real-valued measurable functions on a measurable set $D \subset \mathbb{R}$. Let f be a real-valued measurable functions on D. If $f_n \xrightarrow{\mu} f$ on D, then there exists a subsequence (f_{n_k}) which converges to f a.e. on D.

* * **

Problem 39(An exercise to warn up.)
1. Consider the sequence (f_n) *defined on* $\mathbb{R}$ *by* $f_n = \chi_{[n,n+1]}$, $n \in \mathbb{N}$ *and the function* $f \equiv 0$. *Does* (f_n) *converge to* f *a.e.? a.u.? in measure?*
2. Same questions with $f_n = n\chi_{[0,\frac{1}{n}]}$.

(Note: χ_A is the characteristic function of the set A. Try to write your solution.)

Problem 40
Let (f_n) *be a sequence of extended real-valued measurable functions on* X *and let* f *be an extended real-valued function which is finite a.e. on* X. *Suppose* $\lim_{n\to\infty} f_n = f$ *a.e. on* X. *Let* $\alpha \in [0, \mu(X))$ *be arbitrarily chosen. Show that for every* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $\mu\{X : |f_n - f| < \varepsilon\} \geq \alpha$ *for* $n \geq N$.

Solution
Let Z be a null set such that f is finite on $X \setminus Z$. Since $f_n \to f$ a.e. on X, $f_n \to f$ a.e. on $X \setminus Z$. For every $\varepsilon > 0$ we have[1]

$$\begin{aligned} &\mu(\limsup_{n\to\infty}\{X \setminus Z : |f_n - f| \geq \varepsilon\}) = 0 \\ &\Rightarrow \limsup_{n\to\infty} \mu\{X \setminus Z : |f_n - f| \geq \varepsilon\} = 0 \\ &\Rightarrow \lim_{n\to\infty} \mu\{X \setminus Z : |f_n - f| \geq \varepsilon\} = 0 \end{aligned}$$

The last condition is equivalent to

$$\begin{aligned} &\lim_{n\to\infty} \mu\{X \setminus Z : |f_n - f| < \varepsilon\} = \mu(X \setminus Z) = \mu(X) \\ &\Leftrightarrow \forall \eta > 0, \exists N \in \mathbb{N} : \mu(X) - \mu\{X \setminus Z : |f_n - f| < \varepsilon\} \leq \eta \text{ for all } n \geq N. \end{aligned}$$

Let us take $\eta = \mu(X) - \alpha > 0$. Then we have

$$\exists N \in \mathbb{N} : \mu\{X \setminus Z : |f_n - f| < \varepsilon\} \geq \alpha \text{ for all } n \geq N.$$

Since $\{X : |f_n - f| < \varepsilon\} \supset \{X \setminus Z : |f_n - f| < \varepsilon\}$, so we have

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow \mu(\{X : |f_n - f| < \varepsilon\}) \geq \alpha. \qquad \blacksquare$$

[1]See Problem 11b. We have

$$\mu\big(\limsup_{n\to\infty} E_n\big) \geq \limsup_{n\to\infty} \mu(E_n).$$

Problem 41
(a) Show that the condition

$$\lim_{n\to\infty} \mu\{x \in D: |f_n(x) - f(x)| > 0\} = 0$$

implies that $f_n \xrightarrow{\mu} f$ on D.
(b) Show that the converse is not true.
(c) Show that the condition in (a) implies that for a.e. $x \in D$ we have $f_n(x) = f(x)$ for infinitely many $n \in \mathbb{N}$.

Solution
(a) Given any $\varepsilon > 0$, for every $n \in \mathbb{N}$, let

$$E_n = \{x \in D: |f_n(x) - f(x)| > \varepsilon\}; \quad F_n = \{x \in D: |f_n(x) - f(x)| > 0\}.$$

Then we have for all $n \in \mathbb{N}$

$$\begin{aligned} x \in E_n &\Rightarrow |f_n(x) - f(x)| > \varepsilon \\ &\Rightarrow |f_n(x) - f(x)| > 0 \\ &\Rightarrow x \in F_n. \end{aligned}$$

Consequently, $E_n \subset F_n$ and $\mu(E_n) \le \mu(F_n)$ for all $n \in \mathbb{N}$. By hypothesis, we have that $\lim_{n\to\infty} \mu(F_n) = 0$. This implies that $\lim_{n\to\infty} \mu(E_n) = 0$. Thus, $f_n \xrightarrow{\mu} f$.
(b) The converse of (a) is false.
Consider functions:

$$\begin{aligned} &f_n(x) = \frac{1}{n}, \quad x \in [0,1] \quad n \in \mathbb{N}. \\ &f(x) = 0, \quad x \in [0,1]. \end{aligned}$$

Then $f_n \to f$ (pointwise) on $[0,1]$. By Lebesgue Theorem $f_n \xrightarrow{\mu} f$ on $[0,1]$. But for every $n \in \mathbb{N}$

$$|f_n(x) - f(x)| = \frac{1}{n} > 0, \ \forall x \in [0,1].$$

In other words,

$$\{x \in D: |f_n(x) - f(x)| > 0\} = [0,1].$$

Thus,

$$\lim_{n\to\infty} \mu\{x \in D: |f_n(x) - f(x)| > 0\} = 1 \neq 0.$$

(c) Recall that (Problem 11a)

$$\mu(\liminf_{n\to\infty} E_n) \le \liminf_{n\to\infty} \mu(E_n). \qquad (*)$$

Let $E_n = \{x \in D: \ f_n(x) \neq f(x)\}$ and $E = \liminf_{n\to\infty} E_n$. By (a),

$$\liminf_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(E_n) = 0.$$

Therefore, by $(*)$, $\mu(E) = 0$. By definition, we have

$$E = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} E_k.$$

Hence, $x \notin E$ whenever $x \in E_n^c$ for infinitely many n's, that is $f_n(x) = f(x)$ a.e. in D for infinitely many n's. ■

Problem 42
Suppose $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in D \setminus Z$ with $\mu(Z) = 0$. If $f_n \xrightarrow{\mu} f$ on D , then prove that $f_n \to f$ a.e. on D.

Solution
Let $B = D \setminus Z$. Since $f_n \xrightarrow{\mu} f$ on D, $f_n \xrightarrow{\mu} f$ on B. Then, By Riesz theorem, there exists a sub-sequence (f_{n_k}) of (f_n) such that $f_{n_k} \to f$ a.e. on B.
Let $C = \{x \in B: \ f_{n_k} \nrightarrow f\}$. Then $\mu(C) = 0$ and $f_{n_k} \to f$ on $B \setminus C$.
From $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$, and since $n_k \geq k$, we get $f_k \leq f_{n_k}$ for all $k \in \mathbb{N}$. Therefore

$$|f_k - f| \leq |f_{n_k} - f|.$$

This implies that $f_k \to f$ on $B \setminus C$. Since $B \setminus C = D \setminus (Z \cup C)$ and $\mu(Z \cup C) = 0$, it follows that $f_n \to f$ a.e. on D ■.

Problem 43
Show that if $f_n \xrightarrow{\mu} f$ on D and $g_n \xrightarrow{\mu} g$ on D then $f_n + g_n \xrightarrow{\mu} f + g$ on D.

Solution
Since $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ on D, for every $\varepsilon > 0$,

(4.1) $$\lim_{n\to\infty} \mu\{D: \ |f_n - f| \geq \frac{\varepsilon}{2}\} = 0$$

(4.2) $$\lim_{n\to\infty} \mu\{D: \ |g_n - g| \geq \frac{\varepsilon}{2}\} = 0.$$

Now

$$|(f_n + g_n) - (f + g)| \leq |f_n - f| + |g_n - g|.$$

By the triangle inequality above, if $|(f_n + g_n) - (f + g)| \geq \varepsilon$ is true, then at least one of the two following inequalities must be true:

$$|f_n - f| \geq \frac{\varepsilon}{2} \text{ or } |g_n - g| \geq \frac{\varepsilon}{2}.$$

Hence

$$\{D: \ |(f_n + g_n) - (f + g)| \geq \varepsilon\} \subset \left\{D : |f_n - f| \geq \frac{\varepsilon}{2}\right\} \cup \left\{D : |g_n - g| \geq \frac{\varepsilon}{2}\right\}.$$

Therefore,

$$\mu\{D: \ |(f_n + g_n) - (f + g)| \geq \varepsilon\} \leq \mu\left\{D : |f_n - f| \geq \frac{\varepsilon}{2}\right\} + \mu\left\{D : |g_n - g| \geq \frac{\varepsilon}{2}\right\}.$$

From (4.1) and (4.2) we obtain

$$\lim_{n\to\infty} \mu\{D: \ |(f_n + g_n) - (f + g)| \geq \varepsilon\} = 0.$$

That is, by definition, $f_n + g_n \xrightarrow{\mu} f + g$ on D. ■

Problem 44
Show that if $f_n \xrightarrow{\mu} f$ on D and $g_n \xrightarrow{\mu} g$ on D and $\mu(D) < \infty$, then $f_n g_n \xrightarrow{\mu} fg$ on D.
(Assume that both f_n and g_n are real-valued for every $n \in \mathbb{N}$ so that the multiplication $f_n g_n$ is possible.)

Solution
For every $\varepsilon > 0$ and $\delta > 0$, we want $\mu\{|f_n g_n - fg| \geq \varepsilon\} < \delta$ for n large enough. Notice that

$$(*) \quad |f_n g_n - fg| \leq |f_n g_n - f g_n| + |f g_n - fg| \leq |f_n - f||g_n| + |f||g_n - g|.$$

For any $N \in \mathbb{N}$, let

$$E_N = \{D: \ |f| > N\} \cup \{D: \ |g| > N\}.$$

It is clear that $E_N \supset E_{N+1}$ for every $N \in \mathbb{N}$. Since $\mu(D) < \infty$, we have

$$\lim_{N\to\infty} \mu(E_N) = \mu\left(\bigcap_{N\in\mathbb{N}} E_N\right) = \mu(\varnothing) = 0.$$

It follows that, we can take N large enough to get, for every $\delta > 0$,

$$(**)\qquad \frac{\varepsilon}{2N} < 1 \text{ and } \mu(E_N) < \frac{\delta}{3}.$$

Observe that

$$\{D:\ |g_n| > N+1\} \subset \Big\{D:\ |g_n - g| \geq \frac{\varepsilon}{2N}\Big\} \cup E_N$$

(since $|g_n| \leq |g_n - g| + |g|$). Now if we have

$$|f_n - f| \geq \frac{\varepsilon}{2(N+1)};\ |g_n| > N+1;\ |g_n - g| \geq \frac{\varepsilon}{2N}, \text{ and } |f| > N,$$

then (*) implies

$$\begin{aligned}\{D:\ |f_n g_n - fg| \geq \varepsilon\} \ &\subset\ \Big\{D:\ |f_n - f| \geq \frac{\varepsilon}{2(N+1)}\Big\} \cup E_N \\ &\cup\ \Big\{D:\ |g_n - g| \geq \frac{\varepsilon}{2N}\Big\} \cup \{D:\ |g_n| > N+1\}.\end{aligned}$$

By assumption, given $\varepsilon > 0,\ \delta > 0$, for $n > N$, we have

$$\mu\Big\{D:\ |f_n - f| \geq \frac{\varepsilon}{2(N+1)}\Big\} < \frac{\delta}{3}$$

$$\mu\Big\{D:\ |g_n - g| \geq \frac{\varepsilon}{2N}\Big\} < \frac{\delta}{3}.$$

From these results, from (*), and (**) we get

$$\mu\{D:\ |f_n g_n - fg| \geq \varepsilon\} < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta. \qquad \blacksquare$$

Problem 45
(a) Definition of "Almost uniform convergence" (a.u).
(b) Show that if $f_n \to f$ a.u on D then $f_n \xrightarrow{\mu} f$ on D.
(c) Show that if $f_n \to f$ a.u on D then $f_n \to f$ a.e. on D.

Solution
(a) $\forall \varepsilon > 0, \exists E \subset D$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $D \setminus E$.

(b) Suppose that $f_n \to f$ a.u on D and f_n does not converges to f in measure on D. Then there exists an $\varepsilon_0 > 0$ such that

$$\mu\{x \in D:\ |f_n(x) - f(x)| > \varepsilon_0\} \nrightarrow 0 \text{ as } n \to \infty.$$

We can choose $n_1 < n_2 < ...$ such that

$$\mu\{x \in D : \ |f_{n_k}(x) - f(x)| > \varepsilon_0\} \geq r \ \text{ for some } \ r > 0 \ \text{ and } \ \forall k \in \mathbb{N}.$$

Now since $f_n \to f$ a.u on D,

$$\exists E \subset D \text{ such that } \mu(E) < \frac{r}{2} \text{ and } \ f_n \to f \text{ uniformly on } D \setminus E.$$

Let $C = \{x \in D : \ |f_{n_k}(x) - f(x)| > \varepsilon_0\} \ \ \forall k \in \mathbb{N}$. Then $\mu(C) \geq r$. Since $f_n \to f$ uniformly on $D \setminus E$,

$$\exists N : \ n \geq N \Rightarrow |f_n(x) - f(x)| \leq \varepsilon_0, \ \forall x \in D \setminus E.$$

Thus,

$$C \subset (D \setminus E)^c = E.$$

Hence,

$$0 < r \leq \mu(C) \leq \mu(E) < \frac{r}{2}.$$

This is a contradiction.

(c) Since $f_n \to f$ a.u. on D, for every $n \in \mathbb{N}$, there exists $E_n \subset D$ such that $\mu(E_n) < \frac{1}{n}$ and $f_n \to f$ uniformly on $D \setminus E_n$. Let $E = \bigcap_{n\in\mathbb{N}} E_n$, then $\mu(E) = 0$. Since $f_n \to f$ on $D \setminus E_n$ for every $n \in \mathbb{N}$, $f_n \to f$ on

$$\bigcup_{n\in\mathbb{N}} (D \setminus E_n) = D \setminus \bigcap_{n\in\mathbb{N}} E_n = D \setminus E.$$

Since $\mu(E) = 0$, $f_n \to f$ a.e. on D ■

Chapter 5

Integration of Bounded Functions on Sets of Finite Measure

In this chapter we suppose $\mu(D) < \infty$.

1. Integration of simple functions

Definition 17 *(Lebesgue integral of simple functions)*
Let φ be a simple function on D and $\varphi = \sum_{i=1}^{n} a_i \chi_{D_i}$ be its canonical representation. The Lebesgue integral of φ on D is defined by

$$\int_D \varphi(x)\mu(dx) = \sum_{i=1}^{n} a_i \mu(D_i).$$

We usually use simple notations for the integral of φ:

$$\int_D \varphi d\mu, \quad \int_D \varphi(x)dx \quad or \quad \int_D \varphi.$$

If $\int_D \varphi d\mu < \infty$, then we say that φ is integrable on D.

Proposition 15 *(properties of integral of simple functions)*

1. $\mu(D) = 0 \;\Rightarrow\; \int_D \varphi d\mu = 0$.
2. $\varphi \geq 0,\; E \subset D \;\Rightarrow\; \int_E \varphi d\mu \leq \int_D \varphi d\mu$.
3. $\int_D c\varphi d\mu = c \int_D \varphi d\mu$.
4. $\int_D \varphi d\mu = \sum_{i=1}^{n} \int_{D_i} \varphi d\mu$.
5. $\int_D c\varphi d\mu = c \int_D \varphi d\mu$ *(c is a constant).*
6. $\int_D (\varphi_1 + \varphi_2) d\mu = \int_D \varphi_1 d\mu + \int_D \varphi_2 d\mu$.
7. $\varphi_1 = \varphi_2$ *a.e. on* $D \Rightarrow \int_D \varphi_1 d\mu = \int_D \varphi_2 d\mu$.

2. Integration of bounded functions

Definition 18 *(Lebesgue integral of bounded functions)*
Let f be a bounded real-valued measurable function on D. Let Φ be the collection of all simple functions on D. We define the Lebesgue integral of f on D by

$$\int_D f d\mu = \inf_{\psi \geq f} \int_D \psi d\mu = \sup_{\varphi \leq f} \int_D \varphi d\mu \quad \textit{where} \quad \varphi, \psi \in \Phi.$$

If $\int_D f d\mu < \infty$, then we say that f is integrable on D.

Proposition 16 *(properties of integral of bounded functions)*

1. $\int_D cf d\mu = c \int_D f d\mu$.
2. $\int_D (f+g) d\mu = \int_D f d\mu + \int_D g d\mu$.
3. $f = g$ *a.e. on* $D \Rightarrow \int_D f d\mu = \int_D g d\mu$.
4. $f \leq g$ *on* $D \Rightarrow \int_D f d\mu \leq \int_D g d\mu$.
5. $|f| \leq M$ *on* $D \Rightarrow \left|\int_D f d\mu\right| \leq \int_D |f| d\mu \leq M\mu(D)$.
6. $f \geq 0$ *a.e. on* D *and* $\int_D f d\mu = 0 \Rightarrow f = 0$ *a.e. on* D.
7. *If (D_n) be a disjoint sequence of measurable subset $D_n \subset D$ with $\bigcup_{n\in\mathbb{N}} D_n = D$ then*

$$\int_D f d = \mu \sum_{n\in\mathbb{N}} \int_{D_n} f d\mu.$$

Theorem 5 *(Bounded convergence theorem)*
Suppose that (f_n) is a uniformly bounded sequence of real-valued measurable functions on D, and f is a bounded real-valued measurable function on D. If $f_n \to f$ a.e. on D, then

$$\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0.$$

In particular,

$$\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu.$$

* * **

Problem 46
Let f be an extended real-valued measurable function on a measurable set D. For $M_1, M_2 \in \mathbb{R},\ M_1 < M_2$, let the truncation of f at M_1 and M_2 be defined by

$$g(x) = \begin{cases} M_1 & \text{if } f(x) < M_1 \\ f(x) & \text{if } M_1 \leq f(x) \leq M_2 \\ M_2 & \text{if } f(x) > M_2. \end{cases}$$

Show that g is measurable on D.

Solution
Let $a \in \mathbb{R}$. We need to show that the set $E = \{x \in D: \ g(x) > a\}$ is measurable. There are three cases to consider:

1. If $a \geq M_2$ then $E = \varnothing$ which is measurable.
2. If $a < M_1$ then $E = D$ which is measurable.
3. If $M_1 \leq a < M_2$ then $E = \{x \in D: \ f(x) > a\}$ which is measurable.

Thus, in all three cases E is measurable, so g is measurable. ■

Problem 47
Given a measure space $(X, \mathcal{A}, \mu)$. Let f be a bounded real-valued $\mathcal{A}$-measurable function on $D \in \mathcal{A}$ with $\mu(D) < \infty$. Suppose $|f(x)| \leq M,\ \forall x \in D$ for some constant $M > 0$.
(a) Show that if $\int_D f d\mu = M\mu(D)$, then $f = M$ a.e. on D.
(b) Show that if $f < M$ a.e. on D and if $\mu(D) > 0$, then $\int_D f d\mu < M\mu(D)$.

Solution
(a) For every $n \in \mathbb{N}$, let $E_n = \{x \in D: \ f(x) < M - \frac{1}{n}\}$. Then, since $f \leq M$ on $D \setminus E_n$, we have

$$\begin{aligned} \int_D f d\mu &= \int_{E_n} f d\mu + \int_{D \setminus E_n} f d\mu \\ &\leq \left(M - \frac{1}{n}\right) \mu(E_n) + M\mu(D \setminus E_n). \end{aligned}$$

Since $E_n \subset D$, we have

$$\mu(D \setminus E_n) = \mu(D) - \mu(E_n).$$

Therefore,

$$\begin{aligned} \int_D f d\mu &\leq \left(M - \frac{1}{n}\right)\mu(E_n) + M\mu(D) - M\mu(E_n) \\ &= M\mu(D) - \frac{1}{n}\mu(E_n). \end{aligned}$$

By assumption $\int_D f d\mu = M\mu(D)$, it follows that

$$0 \leq -\frac{1}{n}\mu(E_n) \leq 0, \ \forall n \in \mathbb{N},$$

which implies $\mu(E_n) = 0, \ \forall n \in \mathbb{N}$.
Now let $E = \bigcup_{n=1}^{\infty} E_n$ then $E = \{x \in D : \ f(x) < M\}$. We want to show that $\mu(E) = 0$. We have

$$0 \leq \mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0.$$

Thus, $\mu(E) = 0$. Since $|f| \leq M$, the last result implies $f = M$ a.e. on D.

(b) First we note that $|f| \leq M$ on D implies that $\int_D f d\mu \leq M\mu(D)$. Assume that $\int_D f d\mu = M\mu(D)$. By part (a) we have $f = M$ a.e. on D. This contradicts the fact that $f < M$ a.e. on D. Thus $\int_D f d\mu < M\mu(D)$. ■

Problem 48
Consider a sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined on $[0,1]$ by

$$f_n(x) = \frac{nx}{1+n^2x^2} \quad \text{for} \quad x \in [0,1].$$

(a) Show that (f_n) is uniformly bounded on $[0,1]$ and evaluate

$$\lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\, d\mu.$$

(b) Show that (f_n) does not converge uniformly on $[0,1]$.

Solution
(a) For all $n \in \mathbb{N}$, for all $x \in [0,1]$, we have $1 + n^2x^2 \geq 2nx \geq 0$ and $1 + n^2x^2 > 0$, hence

$$0 \leq f_n(x) = \frac{nx}{1+n^2x^2} \leq \frac{1}{2}.$$

Thus, (f_n) is uniformly bounded on $[0,1]$.
Since each f_n is continuous on $[0,1]$, f is Riemann integrable on $[0,1]$. In this case, Lebesgue integral and Riemann integral on $[0,1]$ coincide:

$$\begin{aligned}\int_{[0,1]} \frac{nx}{1+n^2x^2}\, d\mu &= \int_0^1 \frac{nx}{1+n^2x^2}dx \\ &= \frac{1}{2n}\int_1^{1+n^2} \frac{1}{t}dt \quad (\text{with } \ t=1+n^2x^2) \\ &= \frac{1}{2n}\ln(1+n^2) = \frac{\ln(1+n^2)}{2n}.\end{aligned}$$

Using L'Hospital rule we get $\lim_{x\to\infty}\frac{\ln(1+x^2)}{2x}=0$. Hence,

$$\lim_{n\to\infty}\int_{[0,1]} \frac{nx}{1+n^2x^2}d\mu = 0.$$

(b) For each $x\in[0,1]$,

$$\lim_{n\to\infty}\frac{nx}{1+n^2x^2}=0.$$

Hence, $f_n\to f\equiv 0$ pointwise on $[0,1]$. To show f_n does not converge to $f\equiv 0$ uniformly on $[0,1]$, we find a sequence (x_n) in $[0,1]$ such that $x_n\to 0$ and $f_n(x_n)\not\to f(0)=0$ as $n\to\infty$. Indeed, take $x_n=\frac{1}{n}$. Then $f_n(x)=\frac{1}{2}$. Thus,

$$\lim_{n\to\infty} f_n(x_n)=\frac{1}{2}\neq f(0)=0. \quad \blacksquare$$

Problem 49
Let $(f_n)_{n\in\mathbb{N}}$ and f be extended real-valued measurable functions on $D\in\mathcal{M}_L$ with $\mu(D)<\infty$ and assume that f is real-valued a.e. on D. Show that $f_n\xrightarrow{\mu} f$ on D if and only if

$$\lim_{n\to\infty}\int_D \frac{|f_n-f|}{1+|f_n-f|}d\mu=0.$$

Solution
• Suppose $f_n\xrightarrow{\mu} f$ on D. By definition of convergence in measure, for any $\varepsilon>0$, there exists an $N\in\mathbb{N}$ such that for $n\geq N$,

$$\exists E_n\subset D:\ \mu(E_n)<\frac{\varepsilon}{2} \ \text{ and } \ |f_n-f|<\frac{\varepsilon}{2\mu(D)} \ \text{ on } \ D\setminus E_n.$$

For $n \geq N$ we have

$$(*) \quad \int_D \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu + \int_{D \setminus E_n} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu.$$

Note that for all $n \in \mathbb{N}$, we have $0 \leq \frac{|f_n - f|}{1+|f_n - f|} \leq 1$ on E_n and

$$0 \leq \frac{|f_n - f|}{1 + |f_n - f|} = |f_n - f|\, \frac{1}{1 + |f_n - f|} \leq |f_n - f| \leq \frac{\varepsilon}{2\mu(D)} \quad \text{on} \quad D \setminus E_n.$$

So for $n \geq N$, we can write (*) as

$$\begin{aligned} 0 \leq \int_D \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu &\leq \int_{E_n} 1\, d\mu + \int_{D \setminus E_n} \frac{\varepsilon}{2\mu(D)}\, d\mu \\ &= \mu(E_n) + \frac{\varepsilon}{2\mu(D)}\, \mu(D \setminus E_n) \\ &\leq \mu(E_n) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} \int_D \frac{|f_n - f|}{1+|f_n - f|}\, \mu(dx) = 0$.

- Conversely, suppose $\lim_{n\to\infty} \int_D \frac{|f_n - f|}{1+|f_n - f|} d\mu = 0$. We show $f_n \xrightarrow{\mu} f$ on D. For any $\varepsilon > 0$, for $n \in \mathbb{N}$, let $E_n = \{x \in D : \ |f_n - f| \geq \varepsilon\}$. We have

$$|f_n - f| \geq \varepsilon \ \Rightarrow \ \frac{|f_n - f|}{1 + |f_n - f|} \geq \frac{\varepsilon}{1 + \varepsilon}$$

(since the function $\varphi(x) = \frac{x}{1+x}$, $x > 0$ is increasing).
It follows that

$$0 \leq \int_{E_n} \frac{\varepsilon}{1 + \varepsilon}\, d\mu \leq \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \leq \int_D \frac{|f_n - f|}{1 + |f_n - f|} d\mu.$$

Hence,

$$0 \leq \frac{\varepsilon}{1 + \varepsilon}\, \mu(E_n) \leq \int_D \frac{|f_n - f|}{1 + |f_n - f|} d\mu.$$

Since $\lim_{n\to\infty} \int_D \frac{|f_n - f|}{1+|f_n - f|} d\mu = 0$, $\lim_{n\to\infty} \mu(E_n) = 0$. Thus, $f_n \xrightarrow{\mu} f$ on D. ■

Problem 50
Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let Φ be the set of all extended real-valued $\mathcal{A}$-measurable function on X where we identify functions that are equal a.e. on X. Let

$$\rho(f, g) = \int_X \frac{|f - g|}{1 + |f - g|}\, d\mu \quad \textit{for} \quad f, g \in \Phi.$$

(a) Show that ρ is a metric on Φ.
(b) Show that Φ is complete w.r.t. the metric ρ.

Solution
(a) Note that $\mu(X)$ is finite and $0 \leq \frac{|f-g|}{1+|f-g|} < 1$, so $0 \leq \rho < \infty$.

- $\rho(f,g) = 0 \;\Leftrightarrow\; \int_X \frac{|f-g|}{1+|f-g|}\, d\mu = 0 \;\Leftrightarrow\; f-g=0 \;\Leftrightarrow\; f=g$. (We identify functions that are equal a.e. on X.)
- It is clear that $\rho(f,g) = \rho(g,f)$.
- We make use the fact that the function $\varphi(x) = \frac{x}{1+x},\; x > 0$ is increasing. For $f, g, h \in \Phi$,

$$\begin{aligned}
\frac{|f-h|}{1+|f-h|} &\leq \frac{|f-g|+|g-h|}{1+|f-g|+|g-h|} \\
&= \frac{|f-g|}{1+|f-g|+|g-h|} + \frac{|g-h|}{1+|f-g|+|g-h|} \\
&\leq \frac{|f-g|}{1+|f-g|} + \frac{|g-h|}{1+|g-h|}.
\end{aligned}$$

Integrating over X we get

$$\int_X \frac{|f-h|}{1+|f-h|}\, d\mu \leq \int_X \frac{|f-g|}{1+|f-g|}\, d\mu + \int_X \frac{|g-h|}{1+|g-h|}\, d\mu.$$

That is

$$\rho(f,g) \leq \rho(f,h) + \rho(h,g).$$

Thus, ρ is a metric on Φ.

(b) Let (f_n) be a Cauchy sequence in Φ. We show that there exists an $f \in \Phi$ such that $\rho(f_n, f) \to 0$ as $n \to \infty$.
First we claim that (f_n) is a Cauchy sequence w.r.t. convergence in measure. Let $\eta > 0$. For $n, m \in \mathbb{N}$, define $A_{m,n} = \{X : \; |f_n - f_m| \geq \eta\}$. For every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$(*) \qquad n, m \geq N \;\Rightarrow\; \rho(f_n, f_m) < \varepsilon \frac{\eta}{1+\eta}.$$

While we have that

$$\rho(f_n, f_m) = \int_X \frac{|f_n - f_m|}{1+|f_n - f_m|}\, d\mu \;\geq\; \int_{A_{m,n}} \frac{|f_n - f_m|}{1+|f_n - f_m|}\, d\mu \;\geq\; \frac{\eta}{1+\eta}\, \mu(A_{m,n}).$$

For $n, m \geq N$, from (*) we get

$$\varepsilon \frac{\eta}{1+\eta} > \frac{\eta}{1+\eta}\, \mu(A_{m,n}).$$

This implies that $\mu(A_{m,n}) < \varepsilon$. Thus, (f_n) is Cauchy in measure. We know that if (f_n) is Cauchy in measure then (f_n) converges in measure to some $f \in \Phi$.

Next we prove that $\rho(f_n, f) \to 0$. Since $f_n \xrightarrow{\mu} f$, for any $\varepsilon > 0$ there exists $E \in \mathcal{A}$ and an $N \in \mathbb{N}$ such that

$$\mu(E) < \frac{\varepsilon}{2} \text{ and } |f_n - f| < \frac{\varepsilon}{2\mu(X)} \text{ on } X \setminus E \text{ whenever } n \geq N.$$

On $X \setminus E$, for $n \geq N$, we have

$$\int_{X\setminus E} \frac{|f_n - f|}{1+|f_n - f|}\, d\mu \leq \int_{X\setminus E} |f_n - f| d\mu < \frac{\varepsilon}{2\mu(X)}\, \mu(X \setminus E) \leq \frac{\varepsilon}{2}.$$

On E, for all n, we have

$$\int_E \frac{|f_n - f|}{1+|f_n - f|}\, d\mu \leq \int_E 1\, d\mu = \mu(E) < \frac{\varepsilon}{2}.$$

Hence, for $n \geq N$, we have

$$\rho(f_n, f) = \int_X \frac{|f_n - f|}{1+|f_n - f|}\, d\mu = \int_E \frac{|f_n - f|}{1+|f_n - f|}\, d\mu + \int_{X\setminus E} \frac{|f_n - f|}{1+|f_n - f|}\, d\mu < \varepsilon.$$

Thus, (f_n) converges to $f \in \Phi$. And hence, (Φ, ρ) is complete ■

Problem 51(Bounded convergence theorem under convergence in measure)
Suppose that (f_n) is a uniformly bounded sequence of real-valued measurable functions on D, and f is a bounded real-valued measurable function on D. If $f_n \xrightarrow{w} f$ on D, then

$$\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0.$$

Solution
We will use this fact:
Let (a_n) be a sequence of real numbers. If there exists a real number a such that every subsequence (a_{n_k}) has a subsequence $(a_{n_{k_l}})$ converging to a, then the sequence (a_n) converges to a.
Consider the sequence of real numbers

$$a_n = \int_D |f_n - f| d\mu, \ n \in \mathbb{N}.$$

Take an arbitrary subsequence (a_{n_k}). Consider the sequence (f_{n_k}). Since (f_n) converges to f in measure on D, the subsequence (f_{n_k}) converges to f in measure on D too. By Riesz theorem, there exists a subsequence $(f_{n_{k_l}})$ converging to f a.e. on D. Thus by the bounded convergence theorem, we have

$$\lim_{n\to\infty} \int_D |f_{n_{k_l}} - f| d\mu = 0.$$

That is, the subsequence $(a_{n_{k_l}})$ of the arbitrary subsequence (a_{n_k}) of (a_n) converges to 0. Therefore the sequence (a_n) converges to 0. Thus

$$\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0. \qquad \blacksquare$$

Chapter 6

Integration of Nonnegative Functions

Definition 19 *Let f be a nonnegative extended real-valued measurable function on a measurable $D \subset \mathbb{R}$. We define the Lebesgue integral of f on D by*

$$\int_D f d\mu = \sup_{0 \leq \varphi \leq f} \varphi d\mu,$$

where the supremum is on the collection of all nonnegative simple function φ on D.
If the integral is finite, we say that f is integrable on D.

Proposition 17 *(Properties)*
Let f, f_1 and f_2 be nonnegative extended real-valued measurable functions on D. Then

1. $\int_D f d\mu < \infty \Rightarrow f < \infty$ *a.e. on* D.
2. $\int_D f d\mu = 0 \Rightarrow f = 0$ *a.e. on* D.
3. $D_0 \subset D \Rightarrow \int_{D_0} f d\mu \leq \int_D f d\mu$.
4. $f > 0$ *a.e. on* D *and* $\int_D f d\mu = 0 \Rightarrow \mu(D) = 0$.
5. $f_1 \leq f_2$ *on* $D \Rightarrow \int_D f_1 d\mu \leq \int_D f_2 d\mu$.
6. $f_1 = f_2$ *a.e. on* $D \Rightarrow \int_D f_1 d\mu \leq \int_D f_2 d\mu$.

Theorem 6 *(Monotone convergence theorem)*
Let (f_n) be an increasing sequence of nonnegative extended real-valued measurable functions on D. If $f_n \to f$ on D then

$$\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu.$$

Remark: The conclusion is not true for a decreasing sequence.

Proposition 18 *Let (f_n) be an increasing sequence of nonnegative extended real-valued measurable functions on D. Then we have*

$$\int_D \left(\sum_{n\in\mathbb{N}} f_n\right) d\mu = \sum_{n\in\mathbb{N}} \int_D f_n d\mu.$$

Theorem 7 *(Fatou's Lemma)*
Let (f_n) be a sequence of nonnegative extended real-valued measurable functions on D. Then we have

$$\int_D \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_D f_n d\mu.$$

In particular, if $\lim_{n\to\infty} f_n = f$ exists a.e. on D, then

$$\int_D f d\mu \leq \liminf_{n\to\infty} \int_D f_n d\mu.$$

Proposition 19 *(Uniform absolute continuity of the integral)*
Let f be an integrable nonnegative extended real-valued measurable functions on D. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_E f d\mu < \varepsilon$$

for every measurable $E \subset D$ with $\mu(E) < \delta$.

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Problem 52
Let f_1 and f_2 be nonnegative extended real-valued measurable functions on a measurable set $D \subset \mathbb{R}$. Suppose $f_1 \leq f_2$ and f_1 is integrable on D. Prove that $f_2 - f_1$ is defined a.e. on D and

$$\int_D (f_2 - f_1) d\mu = \int_D f_2 d\mu - \int_D f_1 d\mu.$$

Solution

Since f_1 is integrable on D, f_1 is real-valued a.e. on D. Thus there exists a null set $N \subset D$ such that $0 \leq f_1(x) < \infty$, $\forall x \in D \setminus N$. Then $f_2 - f_1$ is defined on $D \setminus N$. That is $f_2 - f_1$ is defined a.e. on D. On the other hand, since $f_2 = f_1 + (f_2 - f_1)$, we have

$$\int_D f_2 d\mu = \int_D \big[f_1 + (f_2 - f_1)\big] d\mu = \int_D f_1 d\mu + \int_D (f_2 - f_1) d\mu.$$

Since $\int_D f_1 d\mu < \infty$, we have

$$\int_D (f_2 - f_1) d\mu = \int_D f_2 d\mu - \int_D f_1 d\mu. \quad \blacksquare$$

Remark: If $\int_D f_1 d\mu = \infty$, $\int_D f_2 d\mu - \int_D f_1 d\mu$ may have the form $\infty - \infty$.

Problem 53
Let f be a non-negative real-valued measurable function on a measure space $(X, \mathcal{A}, \mu)$. Suppose that $\int_E f d\mu = 0$ for every $E \in \mathcal{A}$. Show that $f = 0$ a.e.

Solution
Since $f \geq 0$, $A = \{x \in X : f(x) > 0\} = \{x \in X : f(x) \neq 0\}$. We shall show that $\mu(A) = 0$.
Let $A_n = \{x \in X : f(x) \geq \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Then $A = \bigcup_{n \in \mathbb{N}} A_n$. Now on A_n we have

$$\begin{aligned} f \geq \frac{1}{n} \quad &\Rightarrow \quad \int_{A_n} f d\mu \geq \frac{1}{n} \mu(A_n) \\ &\Rightarrow \quad \mu(A_n) \leq n \int_{A_n} f d\mu = 0 \ \text{ (by assumption)} \\ &\Rightarrow \quad \mu(A_n) = 0 \ \text{ for every } \ n \in \mathbb{N}. \end{aligned}$$

Thus, $0 \leq \mu(A) \leq \sum_{n \in \mathbb{N}} \mu(A_n) = 0$. Hence, $\mu(A) = 0$. This tells us that $f = 0$ a.e. $\blacksquare$

Problem 54
Let $(f_n : n \in \mathbb{N})$ be a sequence of non-negative real-valued measurable functions on $\mathbb{R}$ such that $f_n \to f$ a.e. on $\mathbb{R}$.
Suppose $\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} f d\mu < \infty$. Show that for each measurable set $E \subset \mathbb{R}$ we have

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu.$$

Solution

Since $g_n = f_n - f_n\chi_E \geq 0,\ n \in \mathbb{N}$ and $f_n \to f$ a.e., we have, by Fatou's lemma,

$$\begin{aligned}
\int_{\mathbb{R}} \lim_{n\to\infty} g_n d\mu &\leq \liminf_{n\to\infty} \int_{\mathbb{R}} g_n d\mu \\
\int_{\mathbb{R}} (f - f\chi_E) d\mu &\leq \liminf_{n\to\infty} \int_{\mathbb{R}} (f_n - f_n\chi_E) d\mu \\
\int_{\mathbb{R}} f d\mu - \int_E f d\mu &\leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu - \limsup_{n\to\infty} \int_E f_n d\mu.
\end{aligned}$$

From the last inequation and assumption we get

$$\int_E f d\mu \geq \limsup_{n\to\infty} \int_E f_n d\mu. \tag{6.1}$$

Let $h_n = f_n - f_n\chi_E \geq 0$. Using the similar calculation, we obtain

$$\int_E f d\mu \leq \liminf_{n\to\infty} \int_E f_n d\mu. \tag{6.2}$$

From (6.1) and (6.2) we have

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu. \quad \blacksquare$$

Problem 55

Given a measure space $(X, \mathcal{A}, \mu)$. Let (f_n) and f be extended real-valued $\mathcal{A}$-measurable functions on $D \in \mathcal{A}$ and assume that f is real-valued a.e. on D. Suppose there exists a sequence of positive numbers (ε_n) such that

1. $\sum_{n\in\mathbb{N}} \varepsilon_n < \infty$.
2. $\int_D |f_n - f|^p d\mu < \varepsilon_n$ *for every* $n \in \mathbb{N}$ *for some fixed* $p \in (0, \infty)$.

Show that the sequence (f_n) converges to f a.e. on D. (Note that no integrability of $f_n, f, |f|^p$ on D is assumed).

Solution

Since $|f_n - f|^p$ is non-negative measurable for every $n \in \mathbb{N}$, the sequence $\left(\sum_{n=1}^N |f_n - f|^p\right)_{N\in\mathbb{N}}$ is an increasing sequence of non-negative measurable functions. By the Monotone Convergence Theorem, we have

$$\int_D \lim_{N\to\infty} \left(\sum_{n=1}^N |f_n - f|^p\right) d\mu = \lim_{N\to\infty} \int_D \sum_{n=1}^N |f_n - f|^p d\mu.$$

Using assumptions we get

$$\begin{aligned}\int_D \sum_{n=1}^{\infty} |f_n - f|^p d\mu &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_D |f_n - f|^p d\mu \\ &= \sum_{n=1}^{\infty} \int_D |f_n - f|^p d\mu \\ &\leq \sum_{n=1}^{\infty} \varepsilon_n < \infty.\end{aligned}$$

This means that the function under the integral symbol in the left hand side is finite a.e. on D. We have

$$\begin{aligned}\sum_{n=1}^{\infty} |f_n - f|^p < \infty \;\; a.e. \;\text{ on }\; D \quad &\Rightarrow \quad \lim_{n\to\infty} |f_n - f|^p = 0 \;\; a.e. \;\text{ on }\; D \\ &\Rightarrow \quad \lim_{n\to\infty} |f_n - f| = 0 \;\; a.e. \;\text{ on }\; D \\ &\Rightarrow \quad f_n \to f \;\; a.e. \;\text{ on }\; D. \quad \blacksquare.\end{aligned}$$

Problem 56
Given a measure space $(X, \mathcal{A}, \mu)$. Let (f_n) and f be extended real-valued measurable functions on $D \in \mathcal{A}$ and assume that f is real-valued a.e. on D. Suppose $\lim_{n\to\infty} \int_D |f_n - f|^p d\mu = 0$ for some fixed $p \in (0, \infty)$. Show that

$$f_n \xrightarrow{\mu} f \quad on \quad D.$$

Solution
Given any $\varepsilon > 0$. For every $n \in \mathbb{N}$, let $A_n = \{D : \; |f_n - f| \geq \varepsilon\}$. Then

$$\begin{aligned}\int_D |f_n - f|^p d\mu &= \int_{A_n} |f_n - f|^p d\mu + \int_{D\setminus A_n} |f_n - f|^p d\mu \\ &\geq \int_{A_n} |f_n - f|^p d\mu \\ &\geq \varepsilon^p \mu(A_n).\end{aligned}$$

Since $\lim_{n\to\infty} \int_D |f_n - f|^p d\mu = 0$, $\lim_{n\to\infty} \mu(A_n) = 0$. This means that

$$f_n \xrightarrow{\mu} f \;\text{ on }\; D. \quad \blacksquare$$

Problem 57
Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be an extended real-valued $\mathcal{A}$-measurable function on X such that $\int_X |f|^p d\mu < \infty$ for some fixed $p \in (0, \infty)$. Show that

$$\lim_{\lambda \to \infty} \lambda^p \mu\{X : |f| \geq \lambda\} = 0.$$

Solution
For $n = 0, 1, 2, ...$, let $E_n = \{D : n \leq |f| < n + 1\}$. Then $E_n \in \mathcal{A}$ and the E_n's are disjoint. Moreover, $X = \bigcup_{n=0}^{\infty} E_n$. We have

$$\infty > \int_X |f|^p d\mu = \sum_{n=0}^{\infty} \int_{E_n} |f|^p d\mu \geq \sum_{n=0}^{\infty} n^p \mu(E_n).$$

Since $\sum_{n=0}^{\infty} n^p \mu(E_n) < \infty$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have

$$\sum_{n=N}^{\infty} n^p \mu(E_n) < \varepsilon.$$

Note that $n^p \geq N^p$ since $p > 0$. So we have

$$N^p \sum_{n=N}^{\infty} \mu(E_n) < \varepsilon.$$

But $\bigcup_{n=N}^{\infty} E_n = \{X : |f| \geq N\}$. So with the above N, we have

$$N^p \mu\left(\bigcup_{n=N}^{\infty} E_n\right) = N^p \mu\{X : |f| \geq N\} < \varepsilon.$$

Thus,

$$\lim_{\lambda \to \infty} \lambda^p \mu\{X : |f| \geq \lambda\} = 0. \quad \blacksquare$$

Problem 58
Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let f be an extended real-valued $\mathcal{A}$-measurable function on X. Show that for every $p \in (0, \infty)$ we have

$$\int_X |f|^p d\mu = \int_{[0,\infty)} p\lambda^{p-1} \mu\{X : |f| > \lambda\} \mu_L(d\lambda). \quad (*)$$

Solution
We may suppose $f \geq 0$ (otherwise we set $g = |f| \geq 0$).
1. If $f = \chi_E, \quad E \in \mathcal{A}$, then

$$\int_X f^p d\mu = \int_X (\chi_E)^p d\mu = \mu(E).$$
$$\int_{[0,\infty)} p\lambda^{p-1}\mu\{X : \ \chi_E > \lambda\}\mu_L(d\lambda) = \int_0^1 p\lambda^{p-1}\mu(E)d\lambda = \mu(E).$$

Thus, the equality $(*)$ holds.

2. If $f = \sum_{i=1}^n a_i\chi_{E_i}$ (simple function), with $a_i \geq 0, \quad E_i \in \mathcal{A}, \quad i = 1, ..., n.$, then the equality $(*)$ holds because of the linearity of the integral.

3. If $f \geq 0$ measurable, then there is a sequence (φ_n) of non-negative measurable simple functions such that $\varphi_n \uparrow f$. By the Monotone Convergence Theorem we have

$$\begin{aligned}
\int_X f^p d\mu &= \lim_{n\to\infty} \int_X \varphi_n^p d\mu \\
&= \lim_{n\to\infty} \int_{[0,\infty)} p\lambda^{p-1}\mu\{X : \ \varphi_n > \lambda\}\mu_L(d\lambda) \\
&= \int_{[0,\infty)} p\lambda^{p-1}\mu\{X : \ f > \lambda\}\mu_L(d\lambda). \qquad \blacksquare
\end{aligned}$$

Notes:

1. $A = \{X : \ \chi_E > \lambda\} = \{x \in X : \ \chi_E(x) > \lambda\}$.
   - If $0 \leq \lambda < 1$ then $A = E$.
   - If $\lambda \geq 1$ then $A = \emptyset$.
2. Why σ-finite measure?

Problem 59
Given a measure space $(X, \mathcal{A}, \mu)$. Let f be a non-negative extended real-valued $\mathcal{A}$-measurable function on $D \in \mathcal{A}$ with $\mu(D) < \infty$.
Let $D_n = \{x \in D : \ f(x) \geq n\}$ for $n \in \mathbb{N}$. Show that

$$\int_D f d\mu < \infty \ \Leftrightarrow \ \sum_{n\in\mathbb{N}} \mu(D_n) < \infty.$$

Solution
From the expression $D_n = \{x \in D : \ f(x) \geq n\}$ with f $\mathcal{A}$-measurable, we deduce that $D_n \in \mathcal{A}$ and

$$D := D_0 \supset D_1 \supset D_2 \supset ... \supset D_n \supset D_{n+1} \supset ...$$

Moreover, all the sets $D_n \setminus D_{n+1} = \{D : \ n \leq f < n+1, \ n \in \mathbb{N}\}$ are disjoint and

$$D = \bigcup_{n \in \mathbb{N}} (D_n \setminus D_{n+1}).$$

It follows that

$$\begin{aligned}
n\mu(D_n \setminus D_{n+1}) &\leq \int_{D_n \setminus D_{n+1}} f d\mu \leq (n+1)\mu(D_n \setminus D_{n+1}) \\
\sum_{n=0}^{\infty} n\mu(D_n \setminus D_{n+1}) &\leq \int_{\bigcup_{n \in \mathbb{N}}(D_n \setminus D_{n+1})} f d\mu \leq \sum_{n=0}^{\infty}(n+1)\mu(D_n \setminus D_{n+1}) \\
\sum_{n=0}^{\infty} n\mu[(D_n) - \mu(D_{n+1})] &\leq \int_D f d\mu \leq \sum_{n=0}^{\infty}(n+1)[\mu(D_n) - \mu(D_{n+1})]. \quad (i)
\end{aligned}$$

Some more calculations:

$$\begin{aligned}
\sum_{n=0}^{\infty} n\mu[(D_n) - \mu(D_{n+1})] &= 1[\mu(D_1) - \mu(D_2)] + 2[\mu(D_2) - \mu(D_3)] + ... \\
&= \sum_{n=1}^{\infty} \mu(D_n),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+1)[\mu(D_n) - \mu(D_{n+1})] &= 1[\mu(D_0) - \mu(D_1)] + 2[\mu(D_1) - \mu(D_2)] + ... \\
&= \mu(D) + \sum_{n=1}^{\infty} \mu(D_n).
\end{aligned}$$

With these, we rewrite (i) as follows

$$\sum_{n=1}^{\infty} \mu(D_n) \leq \int_D f d\mu \leq \mu(D) + \sum_{n=1}^{\infty} \mu(D_n).$$

Since $\mu(D) < \infty$, we have

$$\int_D f d\mu < \infty \ \Leftrightarrow \ \sum_{n \in \mathbb{N}} \mu(D_n) < \infty. \quad \blacksquare$$

Problem 60
Given a measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) < \infty$. Let f be a non-negative extended real-valued $\mathcal{A}$-measurable function on X. Show that f is μ-integrable on X if and only if

$$\sum_{n=0}^{\infty} 2^n \mu\{x \in X : \; f(x) > 2^n\} < \infty.$$

Solution
Let $E_n = \{X : \; f > 2^n\}$ for each $n = 0, 1, 2, ...$ Then it is clear that

$$\begin{aligned}
&E_0 \supset E_1 \supset ... \supset E_n \supset E_{n+1} \supset ... \\
&E_n \setminus E_{n+1} = \{X : \; 2^n < f \leq 2^{n+1}\} \;\; \text{and are disjoint} \\
&X \setminus E_0 = \{X : \; 0 \leq f \leq 1\} \\
&X = (X \setminus E_0) \cup \bigcup_{n=0}^{\infty} (E_n \setminus E_{n+1}).
\end{aligned}$$

Now we have

$$\begin{aligned}
\int_X f d\mu &= \int_{X \setminus E_0} f d\mu + \int_{\bigcup_{n=0}^{\infty}(E_n \setminus E_{n+1})} f d\mu \\
&= \int_{X \setminus E_0} f d\mu + \sum_{n=0}^{\infty} \int_{E_n \setminus E_{n+1}} f d\mu.
\end{aligned}$$

This implies that

$$(6.3) \qquad \sum_{n=0}^{\infty} \int_{E_n \setminus E_{n+1}} f d\mu = \int_X f d\mu - \int_{X \setminus E_0} f d\mu.$$

On the other hand, for $n = 0, 1, 2, ...$, we have

$$2^n \mu(E_n \setminus E_{n+1}) \leq \int_{E_n \setminus E_{n+1}} f d\mu \leq 2^{n+1} \mu(E_n \setminus E_{n+1}).$$

Therefore,

$$\sum_{n=0}^{\infty} 2^n \mu(E_n \setminus E_{n+1}) \leq \sum_{n=0}^{\infty} \int_{E_n \setminus E_{n+1}} f d\mu \leq \sum_{n=0}^{\infty} 2^{n+1} \mu(E_n \setminus E_{n+1}).$$

From (6.3) we obtain

$$\sum_{n=0}^{\infty} 2^n \mu(E_n \setminus E_{n+1}) + \int_{X \setminus E_0} f d\mu \leq \int_X f d\mu \leq \sum_{n=0}^{\infty} 2^{n+1} \mu(E_n \setminus E_{n+1}) + \int_{X \setminus E_0} f d\mu.$$

Since

$$0 \leq \int_{X \setminus E_0} f d\mu \leq \mu(X \setminus E_0) \leq \mu(X) < \infty,$$

we get

$$\text{(6.4)} \qquad \sum_{n=0}^{\infty} 2^n \mu(E_n \setminus E_{n+1}) \leq \int_X f d\mu \leq \sum_{n=0}^{\infty} 2^{n+1} \mu(E_n \setminus E_{n+1}) + \mu(X).$$

Some more calculations:

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^n \mu(E_n \setminus E_{n+1}) &= \sum_{n=0}^{\infty} 2^n [\mu(E_n) - \mu(E_{n+1})] \\
&= \mu(E_0) - \mu(E_1) + 2[\mu(E_1) - \mu(E_2)] + 4[\mu(E_2) - \mu(E_3)] + ... \\
&= \mu(E_0) + \mu(E_1) + 2\mu(E_2) + 4\mu(E_3) + ... \\
&= \frac{1}{2}\mu(E_0) + \frac{1}{2} \sum_{n=0}^{\infty} 2^n \mu(E_n),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^{n+1} \mu(E_n \setminus E_{n+1}) &= \sum_{n=0}^{\infty} 2^{n+1} [\mu(E_n) - \mu(E_{n+1})] \\
&= 2[\mu(E_0) - \mu(E_1)] + 4[\mu(E_1) - \mu(E_2)] + 8[\mu(E_2) - \mu(E_3)] + ... \\
&= \mu(E_0) + [\mu(E_0) + 2\mu(E_1) + 4\mu(E_2) + 8\mu(E_3) + ...] \\
&= \mu(E_0) + \sum_{n=0}^{\infty} 2^n \mu(E_n).
\end{aligned}$$

With these, we rewrite (6.4) as follows

$$\frac{1}{2}\mu(E_0) + \frac{1}{2} \sum_{n=0}^{\infty} 2^n \mu(E_n) \leq \int_X f d\mu \leq \mu(E_0) + \sum_{n=0}^{\infty} 2^n \mu(E_n) + \mu(X).$$

This implies that

$$\frac{1}{2} \sum_{n=0}^{\infty} 2^n \mu(E_n) \leq \int_X f d\mu \leq \sum_{n=0}^{\infty} 2^n \mu(E_n) + 2\mu(X).$$

Since $\mu(X) < \infty$, we have

$$\int_X f d\mu < \infty \Leftrightarrow \sum_{n=0}^{\infty} 2^n \mu\{x \in X : f(x) > 2^n\} < \infty. \quad \blacksquare$$

Problem 61
(a) Let $\{c_{n,i} : n, i \in \mathbb{N}\}$ *be an array of non-negative extended real numbers. Show that*

$$\liminf_{n\to\infty} \sum_{i\in\mathbb{N}} c_{n,i} \geq \sum_{i\in\mathbb{N}} \liminf_{n\to\infty} c_{n,i}.$$

(b) Show that if $(c_{n,i} : n \in \mathbb{N})$ *is an increasing sequence for each* $i \in \mathbb{N}$*, then*

$$\lim_{n\to\infty} \sum_{i\in\mathbb{N}} c_{n,i} = \sum_{i\in\mathbb{N}} \lim_{n\to\infty} c_{n,i}.$$

Solution
(a) Let $\nu : \mathbb{N} \to [0, \infty]$ denote the counting measure. Consider the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$. It is a measure space in which every $A \subset \mathbb{N}$ is measurable. Let $i \mapsto b(i)$ be any function on $\mathbb{N}$. Then

$$\int_{\mathbb{N}} b d\nu = \sum_{i\in\mathbb{N}} b(i).$$

For the array $\{c_{n,i}\}$, for each $i \in \mathbb{N}$, we can write $c_{n,i} = c_n(i),\ n \in \mathbb{N}$. Then c_n is a non-negative ν-measurable function defined on $\mathbb{N}$. By Fatou's lemma,

$$\int_{\mathbb{N}} \liminf_{n\to\infty} c_n d\nu \leq \liminf_{n\to\infty} \int_{\mathbb{N}} c_n d\nu,$$

that is

$$\sum_{i\in\mathbb{N}} \liminf_{n\to\infty} c_{n,i} \leq \liminf_{n\to\infty} \sum_{i\in\mathbb{N}} c_{n,i}.$$

(b) If $(c_{n,i} : n \in \mathbb{N})$ is an increasing sequence for each $i \in \mathbb{N}$, then the sequence of functions (c_n) is non-negative increasing. By the Monotone Convergence Theorem we have

$$\lim_{n\to\infty} \int_{\mathbb{N}} c_n(i) d\nu = \int_{\mathbb{N}} \lim_{n\to\infty} c_n(i) d\nu,$$

that is

$$\lim_{n\to\infty} \sum_{i\in\mathbb{N}} c_{n,i} = \sum_{i\in\mathbb{N}} \lim_{n\to\infty} c_{n,i}. \quad \blacksquare$$

Chapter 7

Integration of Measurable Functions

Given a measure space $(X, \mathcal{A}, \mu)$. Let f be a measurable function on a set $D \in \mathcal{A}$. We define the positive and negative parts of f by

$$f^+ := \max\{f, 0\} \quad \text{and} \quad f^- := \max\{-f, 0\}.$$

Then we have

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$

Definition 20 *Let f be an extended real-valued measurable function on D. The function f is said to be integrable on D if f^+ and f^- are both integrable on D. In this case we define*

$$\int_D f d\mu = \int_D f^+ d\mu - \int_D f^- d\mu.$$

Proposition 20 *(Properties)*

1. *f is integrable on D if and only if $|f|$ is integrable on D.*
2. *If f is integrable on D then cf is integrable on D, and we have $\int_D cf d\mu = c\int_D f d\mu$, where c is a constant in $\mathbb{R}$.*
3. *If f and g are integrable on D then $f+g$ are integrable on D, and we have $\int_D (f+g)d\mu = \int_D f d\mu + \int_D g d\mu$.*
4. *$f \leq g \Rightarrow \int_D f d\mu \leq \int_D g d\mu$.*
5. *If f is integrable on D then $|f| < \infty$ a.e. on D, that is, f is real-valued a.e. on D.*
6. *If $\{D_1, ..., D_n\}$ is a disjoint collection in $\mathcal{A}$, then*

$$\int_{\bigcup_{i=1}^n D_i} f d\mu = \sum_{i=1}^n \int_{D_i} f d\mu.$$

Theorem 8 *(generalized monotone convergence theorem)*
Let (f_n) be a sequence of integrable extended real-valued functions on D.
1. If (f_n) is increasing and there is a extended real-valued measurable function g such that $f_n \geq g$ for every $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} f_n d\mu = \int_D g d\mu.$$

2. If (f_n) is decreasing and there is a extended real-valued measurable function g such that $f_n \leq g$ for every $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} f_n d\mu = \int_D g d\mu.$$

Theorem 9 *(Lebesgue dominated convergence theorem theorem - D.C.T)*
Let (f_n) be a sequence of integrable extended real-valued functions on D and g be an integrable nonnegative extended real-valued function on D such that $|f_n| \leq g$ on D for every $n \in \mathbb{N}$. If $\lim_{n\to\infty} f_n = f$ exists a.e. on D, then f is integrable on D and

$$\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu \quad \text{and} \quad \lim_{n\to\infty} \int_D |f_n - f| d\mu = 0.$$

* * **

Problem 62
Prove this statement:
Let f be extended real-valued measurable function on a measurable set D. If f is integrable on D, then the set $\{D : f \neq 0\}$ is a σ-finite set.

Solution
For every $n \in \mathbb{N}$ set

$$D_n = \Big\{x \in D : |f(x)| \geq \frac{1}{n}\Big\}.$$

Then we have

$$\{x \in D : f(x) \neq 0\} = \{x \in D : |f(x)| > 0\} = \bigcup_{n\in\mathbb{N}} D_n.$$

Now for each $n \in \mathbb{N}$ we have

$$\frac{1}{n}\mu(D_n) \leq \int_{D_n} |f| d\mu \leq \int_D |f| d\mu < \infty.$$

Thus

$$\mu(D_n) = \mu < \infty, \ \forall n \in \mathbb{N},$$

that is, the set $\{x \in D : f(x) \neq 0\}$ is σ-finite. ■

Problem 63
Let f be extended real-valued measurable function on a measurable set D. If (E_n) is an increasing sequence of measurable sets such that $\lim_{n\to\infty} E_n = D$, then

$$\int_D f d\mu = \lim_{n\to\infty} \int_{E_n} f d\mu.$$

Solution
Since (E_n) is an increasing sequence with limit D, so by definition, we have

$$D = \bigcup_{n=1}^{\infty} E_n.$$

Let

$$D_1 = E_1 \quad \text{and} \quad D_n = E_n \setminus E_{n+1}, \quad n \geq 2.$$

Then $\{D_1, D_2, ...\}$ is a disjoint collection of measurable sets, and we have

$$\bigcup_{i=1}^{n} D_i = E_n \quad \text{and} \quad \bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} E_n = D.$$

Hence

$$\begin{aligned}\int_D f d\mu &= \sum_{n=1}^{\infty} \int_{D_n} f d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{D_i} f d\mu \\ &= \lim_{n\to\infty} \int_{\bigcup_{i=1}^n D_i} f d\mu = \lim_{n\to\infty} \int_{E_n} f d\mu. \quad \blacksquare\end{aligned}$$

Problem 64
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f and g be extended real-valued measurable functions on X. Suppose that f and g are integrable on X and $\int_E f d\mu = \int_E g d\mu$ for every $E \in \mathcal{A}$. Show that $f = g$ a.e. on X.

Solution
• Case 1: f and g are two real-valued integrable functions on X.
Assume that the statement $f = g$ *a.e. on* X is false. Then at least one of the two

sets $E = \{X : f < g\}$ and $F = \{X : f > g\}$ has a positive measure. Consider the case $\mu(E) > 0$. Now since both f and g are real-valued, we have

$$E = \bigcup_{k\in\mathbb{N}} E_k \text{ where } E_k = E = \Big\{X : g - f \geq \frac{1}{k}\Big\}.$$

Then $0 < \mu(E) \leq \sum_{k\in\mathbb{N}} \mu(E_k)$. Thus there exists $k_0 \in \mathbb{N}$ such that $\mu(E_{k_0}) > 0$, so that

$$\int_{E_{k_0}} (g - f)d\mu \geq \frac{1}{k_0}\mu(E_{k_0}) > 0.$$

Therefore

$$\int_{E_{k_0}} gd\mu \geq \int_{E_{k_0}} fd\mu + \frac{1}{k_0}\mu(E_{k_0}) > \int_{E_{k_0}} fd\mu.$$

This is a contradiction. Thus $\mu(E) = 0$. Similarly, $\mu(F) = 0$. This shows that $f = g$ a.e. on X.

• Case 2: General case, where f and g are two extended real-valued integrable functions on X. The integrability of f and g implies that f and g are real-valued a.e. on X. Thus there exists a null set $N \subset X$ such that f and g are real-valued on $X \setminus N$. Set

$$\bar{f} = \begin{cases} f & \text{on } X \setminus N, \\ 0 & \text{on } N. \end{cases} \qquad \text{and} \qquad \bar{g} = \begin{cases} g & \text{on } X \setminus N, \\ 0 & \text{on } N. \end{cases}$$

Then $\bar{f}$ and $\bar{g}$ are real-valued on X, and so on every $E \in \mathcal{A}$ we have

$$\int_E \bar{f}d\mu = \int_E fd\mu = \int_E \bar{g}d\mu = \int_E gd\mu.$$

By the first part of the proof, we have $\bar{f} = \bar{g}$ a.e. on X. Since $\bar{f} = f$ a.e. on X and $\bar{g} = g$ a.e. on X, we deduce that

$$f = g \text{ a.e. on } X. \qquad \blacksquare$$

Problem 65
Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let f, g be extended real-valued measurable functions on X. Show that if $\int_E fd\mu = \int_E gd\mu$ for every $E \in \mathcal{A}$ then $f = g$ a.e. on X. (Note that the integrability of f and g is not assumed.)

Solution
The space $(X, \mathcal{A}, \mu)$ is σ-finite :

$$X = \bigcup_{n\in\mathbb{N}} X_n,\ \mu(X_n) < \infty,\ \ \forall n \in \mathbb{N}\ \text{ and }\ \{X_n :\ n \in \mathbb{N}\}\ \text{ are disjoint.}$$

To show $f = g$ a.e. on X it suffices to show $f = g$ a.e. on each X_n (since countable union of null sets is a null set).
Assume that the conclusion is false, that is if $E = \{X_n :\ f < g\}$ and $F = \{X_n : f > g\}$ then at least one of the two sets has a positive measure. Without lost of generality, we may assume $\mu(E) > 0$.
Now, E is composed of three disjoint sets:

$$\begin{aligned} E^{(1)} &= \{X_n :\ -\infty < f < g < \infty\},\\ E^{(2)} &= \{X_n :\ -\infty < f < g = \infty\},\\ E^{(3)} &= \{X_n :\ -\infty = f < g < \infty\}. \end{aligned}$$

Since $\mu(E) > 0$, at least one of these sets has a positive measure.

1. $\mu(E^{(1)}) > 0$. Let

$$E^{(1)}_{m,k,l} = \{X_n :\ -m \leq f\ ;\ f + \frac{1}{k} \leq g\ ;\ g \leq l\}.$$

Then

$$E^{(1)} = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} E^{(1)}_{m,k,l}.$$

By assumption and the subadditivity of μ we have

$$0 < \mu(E^{(1)} \leq \sum_{m,k,l\in\mathbb{N}} \mu(E^{(1)}_{m,k,l}).$$

This implies that there are some $m_0, k_0, l_0 \in \mathbb{N}$ such that

$$\mu(E_{m_0,k_0,l_0}) > 0.$$

Let $E^* = E_{m_0,k_0,l_0}$ then we have

$$\int_{E^*} (g - f)d\mu \geq \frac{1}{k_0}\mu(E^*) > 0\ \text{ so }\ \int_{E^*} g d\mu > \int_{E^*} f d\mu.$$

This is a contradiction.

2. $\mu(E^{(2)}) > 0$. Let
$$E_l^{(2)} = \{X_n : \ -\infty < f \leq l; \ g = \infty\}.$$
Then
$$E^{(2)} = \bigcup_{l\in\mathbb{N}} E_l^{(2)}.$$
By assumption and the subadditivity of μ we have
$$0 < \mu(E^{(2)}) \leq \sum_{l\in\mathbb{N}} \mu(E_l^{(2)}).$$
This implies that there is some $l_0 \in \mathbb{N}$ such that
$$\mu(E_{l_0}^{(2)}) > 0.$$
Let $E^{**} = E_{l_0}^{(2)}$. Then
$$\int_{E^{**}} g d\mu = \infty > \int_{E^{**}} f d\mu.$$
This contradicts the assumption that $\int_E f d\mu = \int_E g d\mu$ for every $E \in \mathcal{A}$.

3. $\mu(E^{(3)}) > 0$. Let
$$E_m^{(2)} = \{X_n : \ -\infty = f; \ -m \leq g < \infty\}.$$
Then
$$E^{(3)} = \bigcup_{m\in\mathbb{N}} E_m^{(3)}.$$
By assumption and the subadditivity of μ we have
$$0 < \mu(E^{(3)}) \leq \sum_{m\in\mathbb{N}} \mu(E_m^{(3)}).$$
This implies that there is some $m_0 \in \mathbb{N}$ such that
$$\mu(E_{m_0}^{(3)}) > 0.$$
Let $E^{***} = E_{m_0}^{(3)}$. Then
$$\int_{E^{***}} g d\mu \geq -m\mu(E^{***}) > -\infty = \int_{E^{***}} f d\mu :$$
This contradicts the assumption.

Thus, $\mu(E) = 0$. Similarly, we get $\mu(F) = 0$. That is $f = g$ a.e. on X. ■.

Problem 66

Given a measure space $(X, \mathcal{A}, \mu)$. Let f be extended real-valued measurable and integrable function on X.

1. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{A}$ with $\mu(A) < \delta$ then

$$\left| \int_A f d\mu \right| < \varepsilon.$$

2. Let (E_n) be a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(E_n) = 0$. Show that $\lim_{n\to\infty} \int_{E_n} f d\mu = 0$.

Solution

1. For every $n \in \mathbb{N}$, set

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{otherwise.} \end{cases}$$

Then the sequence (f_n) is increasing. Each f_n is bounded and $f_n \to f$ pointwise. By the Monotone Convergence Theorem,

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } \left| \int_X f_N d\mu - \int_X f d\mu \right| < \frac{\varepsilon}{2}.$$

Take $\delta = \frac{\varepsilon}{2N}$. If $\mu(A) < \delta$, we have

$$\begin{aligned} \left| \int_A f d\mu \right| &\leq \left| \int_A (f_N - f) d\mu \right| + \left| \int_A f_N d\mu \right| \\ &\leq \left| \int_X (f_N - f) d\mu \right| + N\mu(A) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\delta}\delta = \varepsilon. \end{aligned}$$

2. Since $\lim_{n\to\infty} \mu(E_n) = 0$, with ε and δ as above, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $\mu(E_n) < \delta$. Then we have

$$\left| \int_{E_n} f d\mu \right| < \varepsilon.$$

This shows that $\lim_{n\to\infty} \int_{E_n} f d\mu = 0$. ■

Problem 67
Given a measure space $(X, \mathcal{A}, \mu)$. Let f be extended real-valued $\mathcal{A}$-measurable and integrable function on X. Let $E_n = \{x \in X : |f(x)| \geq n\}$ for $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} \mu(E_n) = 0$.

Solution
First we note that $X = E_0$. For each $n \in \mathbb{N}$, we have

$$E_n \setminus E_{n+1} = \{X : n \leq |f| < n+1\}.$$

Moreover, the collection $\{E_n \setminus E_{n+1} : n \in \mathbb{N}\} \subset \mathcal{A}$ consists of measurable disjoint sets and

$$\bigcup_{n=0}^{\infty}(E_n \setminus E_{n+1}) = X.$$

By the integrability of f we have

$$\infty > \int_X |f| d\mu = \sum_{n=0}^{\infty} \int_{E_n \setminus E_{n+1}} |f| d\mu \geq \sum_{n=0}^{\infty} n\mu(E_n \setminus E_{n+1}).$$

Some more calculations for the last summation:

$$\begin{aligned}
\sum_{n=0}^{\infty} n\mu(E_n \setminus E_{n+1}) &= \sum_{n=0}^{\infty} n[\mu(E_n) - \mu(E_{n+1})] \\
&= \mu(E_1) - \mu(E_2) + 2[\mu(E_2) - \mu(E_3)] + 3[\mu(E_3) - \mu(E_4)] + ... \\
&= \sum_{n=1}^{\infty} \mu(E_n) < \infty.
\end{aligned}$$

Since the series $\sum_{n=1}^{\infty} \mu(E_n)$ converges, $\lim_{n\to\infty} \mu(E_n) = 0$. ■

Problem 68
Let $(X, \mathcal{A}, \mu)$ be a measure space.
(a) Let $\{E_n : n \in \mathbb{N}\}$ be a disjoint collection in $\mathcal{A}$. Let f be an extended real-valued $\mathcal{A}$-measurable function defined on $\bigcup_{n\in\mathbb{N}} E_n$. If f is integrable on E_n for every $n \in \mathbb{N}$, does $\int_{\bigcup_{n\in\mathbb{N}} E_n} f d\mu$ exist?
(b) Let $(F_n : n \in \mathbb{N})$ be an increasing sequence in $\mathcal{A}$. Let f be an extended real-valued $\mathcal{A}$-measurable function defined on $\bigcup_{n\in\mathbb{N}} F_n$. Suppose f is integrable on E_n for every $n \in \mathbb{N}$ and moreover $\lim_{n\to\infty} \int_{F_n} f d\mu$ exists in $\mathbb{R}$. Does $\int_{\bigcup_{n\in\mathbb{N}} F_n} f d\mu$ exist?

Solution
(a) NO.

$$X=[1,\infty),\quad E_n=[n,n+1),\quad n=1,2,...,\{E_n\}\ \text{ disjoint.}$$
$$\mathcal{A}=\mathcal{M}_L,\quad \mu_L.$$
$$X=\bigcup_{n\in\mathbb{N}}E_n,\quad f(x)=1,\ \forall x\in X.$$
$$\int_{E_n}f\,d\mu=1,\ \forall n\in\mathbb{N},\quad \int_{\bigcup_{n\in\mathbb{N}}E_n}f\,d\mu=\int_{[1,\infty)}1\,d\mu=\infty.$$

(b) NO.

$$X=\mathbb{R},\quad F_n=(-n,n),\quad n=1,2,...,\quad (F_n:\ \ n\in\mathbb{N})\ \text{ increasing}$$
$$\mathcal{A}=\mathcal{M}_L,\quad \mu_L.$$
$$X=\bigcup_{n\in\mathbb{N}}F_n,\quad f(x)=1\text{ for }\ x\geq 0,\quad f(x)=-1\text{ for }\ x<0$$
$$\int_{F_n}f\,d\mu=\int_{(-n,0)}(-1)\,d\mu+\int_{[0,n)}1\,d\mu=0\ \Rightarrow\ \lim_{n\to\infty}\int_{F_n}f\,d\mu=0$$
$$\int_{\bigcup_{n\in\mathbb{N}}F_n}f\,d\mu=\int_{\mathbb{R}}f\,d\mu=\int_{(-\infty,0)}(-1)\,d\mu+\int_{(0,\infty)}1\,d\mu\ \text{ does not exist.}\quad \blacksquare$$

Problem 69
Let f is a real-valued uniformly continuous function on $[0,\infty)$. Show that if f is Lebesgue integrable on $[0,\infty)$, then

$$\lim_{x\to\infty}f(x)=0.$$

Solution
Suppose NOT. Then there exists $\varepsilon>0$ such that for each $n\in\mathbb{N}$, there is $x_n>n$ such that $|f(x_n)|\geq\varepsilon$. W.L.O.G. we may choose (x_n) such that

$$x_{n+1}>x_n+1\ \text{ for all }\ n\in\mathbb{N}.$$

Since f is uniformly continuous on $[0,\infty)$, with the above ε,

$$\exists\delta\in(0,\frac{1}{2}):\ |x-y|<\delta\ \Rightarrow\ |f(x)-f(y)|<\frac{\varepsilon}{2}.$$

In particular, for $x \in I_n = (x_n - \delta, x_n + \delta)$, we have

$$|f(x_n) - f(x)| < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}.$$

This implies

$$|f(x_n)| - |f(x)| < \frac{\varepsilon}{2} \;\Rightarrow\; |f(x)| > |f(x_n)| - \frac{\varepsilon}{2} \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Since $x_{n+1} - x_n > 1$ and $0 < \delta < \frac{1}{2}$, $I_n \cap I_{n+1} = \emptyset$. Moreover, $\bigcup_{n=1}^{\infty} I_n \subset [0, \infty)$. By assumption, f is integrable on $[0, \infty)$, so we have

$$\infty > \int_{[0,\infty)} f d\mu \geq \sum_{n=1}^{\infty} \int_{I_n} f d\mu > \sum_{n=1}^{\infty} \int_{I_n} \frac{\varepsilon}{2} d\mu = \infty.$$

This is a contradiction. Thus,

$$\lim_{x\to\infty} f(x) = 0. \quad \blacksquare$$

Problem 70

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(f_n)_{n\in\mathbb{N}}$, and f, g be extended real-valued $\mathcal{A}$-measurable and integrable functions on $D \in \mathcal{A}$. Suppose that

1. $\lim_{n\to\infty} f_n = f$ *a.e. on* D.
2. $\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu$.
3. *either* $f_n \geq g$ *on* D *for all* $n \in \mathbb{N}$ *or* $f_n \leq g$ *on* D *for all* $n \in \mathbb{N}$.

Show that, for every $E \in \mathcal{A}$ and $E \subset D$, we have

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu.$$

Solution

(a) First we solve the problem in the case the condition 3. is replaced by $f_n \geq 0$ on D for all $n \in \mathbb{N}$.
Let $h_n = f_n - f_n\chi_E$ for every $E \in \mathcal{A}$ and $E \subset D$. Then $h_n \geq 0$ and $\mathcal{A}$-measurable

and integrable on D. Applying Fatou's lemma to h_n and using assumptions, we get

$$\begin{aligned}\int_D f d\mu - \int_E f d\mu = \int_D (f - f\chi_E) d\mu &\leq \liminf_{n\to\infty} \int_D (f_n - f_n\chi_E) d\mu \\ &= \lim_{n\to\infty} \int_D f_n d\mu - \limsup_{n\to\infty} \int_D f_n\chi_E d\mu \\ &= \int_D f d\mu - \limsup_{n\to\infty} \int_E f_n d\mu.\end{aligned}$$

Since f is integrable on D, $\int_D f d\mu < \infty$. From the last inequality we obtain,

$$(*) \qquad \limsup_{n\to\infty} \int_E f_n d\mu \leq \int_E f d\mu.$$

Let $k_n = f_n + f_n\chi_E$ for every $E \in \mathfrak{A}$ and $E \subset D$. Using the same way as in the previous paragraph, we get

$$(**) \qquad \int_E f d\mu \leq \liminf_{n\to\infty} \int_E f_n d\mu.$$

From (*) and (**) we get

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu. \quad \blacksquare$$

Next we are coming back to the problem. Assume $f_n \geq g$ on D for all $n \in \mathbb{N}$. Let $\varphi_n = f_n - g$. Using the above result for $\varphi_n \geq 0$ we get

$$\lim_{n\to\infty} \int_E \varphi_n d\mu = \int_E \varphi d\mu.$$

That is

$$\lim_{n\to\infty} \int_E (f_n - g) d\mu = \int_E (f - g) d\mu$$

$$\lim_{n\to\infty} \int_E f_n d\mu - \int_E g d\mu = \int_E f d\mu - \int_E g d\mu.$$

Since g is integrable on E, $\int_E g d\mu < \infty$. Thus, we have

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu. \quad \blacksquare$$

Problem 71(An extension of the Dominated Convergence Theorem)
Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$, and f, g be extended real-valued $\mathcal{A}$-measurable functions on $D \in \mathcal{A}$. Suppose that

1. $\lim_{n\to\infty} f_n = f$ *and* $\lim_{n\to\infty} g_n = g$ *a.e. on* D.
2. (g_n) *and* g *are all integrable on* D *and* $\lim_{n\to\infty} \int_D g_n d\mu = \int_D g d\mu$.
3. $|f_n| \leq g_n$ *on* D *for every* $n \in \mathbb{N}$.

Prove that f *is integrable on* D *and* $\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu$.

Solution
Consider the sequence $(g_n - f_n)$. Since $|f_n| \leq g_n$, and (f_n) and (g_n) are sequences of measurable functions, the sequence $(g_n - f_n)$ consists of non-negative measurable functions. Using the Fatou's lemma we have

$$\begin{aligned}
&\int_D \liminf_{n\to\infty}(g_n - f_n)d\mu \leq \liminf_{n\to\infty} \int_D (g_n - f_n)d\mu \\
&\int_D \lim_{n\to\infty}(g_n - f_n)d\mu \leq \lim_{n\to\infty} \int_D g_n d\mu - \limsup_{n\to\infty} \int_D f_n d\mu \\
&\int_D g d\mu - \int_D f d\mu \leq \int_D g d\mu - \limsup_{n\to\infty} \int_D f_n d\mu \\
&\int_D f d\mu \geq \limsup_{n\to\infty} \int_D f_n d\mu. \quad (*) \quad (\text{since } \int_D g d\mu < \infty).
\end{aligned}$$

Using the same process for the sequence $(g_n + f_n)$, we have

$$\int_D f d\mu \leq \liminf_{n\to\infty} \int_D f_n d\mu. \quad (**).$$

From (*) and (**) we obtain

$$\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu.$$

The fact that f is integrable comes from g_n is integrable:

$$\begin{aligned}
|f_n| \leq g_n \quad &\Rightarrow \quad \int_D f_n d\mu \leq \int_D g_n d\mu < \infty \\
&\Rightarrow \quad \int_D f d\mu < \infty. \quad \blacksquare
\end{aligned}$$

Problem 72
Given a measure space $(X, \mathcal{A}, \mu)$. *Let* $(f_n)_{n\in\mathbb{N}}$ *and* f *be extended real-valued* $\mathcal{A}$*-measurable and integrabe functions on* $D \in \mathcal{A}$. *Suppose that*

$$\lim_{n\to\infty} f_n = f \ a.e. \ \ on \ \ D.$$

(a) Show that if $\lim_{n\to\infty} \int_D |f_n| d\mu = \int_D |f| d\mu$, *then* $\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu$.
(b) Show that the converse of (a) is false by constructing a counter example.

Solution
(a) We will use Problem 71 for

$$g_n = 2(|f_n| + |g_n|) \ \text{ and } \ h_n = |f_n - f| + |f_n| - |f|, \quad n \in \mathbb{N}.$$

We have

$$\begin{aligned} &h_n \to 0 \ \ a.e. \ \text{ on } \ D, \\ &g_n \to 4|f| \ \ a.e. \ \text{ on } \ D, \\ &|h_n| = h_n \leq 2|f_n| \leq g_n, \\ &\lim_{n\to\infty} \int_D g_n d\mu = 2 \lim_{n\to\infty} \int_D |f_n| d\mu + 2 \int_D |f| d\mu = \int_D 4|f| d\mu. \end{aligned}$$

So all conditions of Problem 71 are satisfied. Therefore,

$$\begin{aligned} &\lim_{n\to\infty} \int_D h_n d\mu = \int_D h d\mu = 0 \ \ (h = 0). \\ &\lim_{n\to\infty} \int_D |f_n - f| d\mu + \lim_{n\to\infty} \int_D |f_n| d\mu - \int_D |f| d\mu = 0. \end{aligned}$$

Since $\lim_{n\to\infty} \int_D |f_n| d\mu - \int_D |f| d\mu = 0$ by assumption, we have

$$\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0.$$

This implies that

$$\lim_{n\to\infty} \left| \int_D f_n d\mu - \int_D f d\mu \right| = 0.$$

Hence, $\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu$.
(b) We will give an example showing that *it is not true* that

$$\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu \ \Rightarrow \ \lim_{n\to\infty} \int_D |f_n| d\mu = \int_D |f| d\mu.$$

$$f_n(x) = \begin{cases} n & \text{if } 0 \le x < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \le x \le 1 - \frac{1}{n} \\ -n & \text{if } 1 - \frac{1}{n} < x \le 1. \end{cases}$$

And so

$$|f_n|(x) = \begin{cases} n & \text{if } 0 \le x < \frac{1}{n} \textit{ or } 1 - \frac{1}{n} < x \le 1 \\ 0 & \text{if } \frac{1}{n} \le x \le 1 - \frac{1}{n}. \end{cases}$$

Then we have

$$f_n \to 0 \equiv 0 \text{ and } \int_{[0,1]} f_n d\mu = 0 \to 0 = \int_{[0,1]} 0 d\mu$$

while

$$\int_{[0,1]} |f_n| d\mu = 2 \to 2 \ne 0. \quad \blacksquare$$

Problem 73

Given a measure space $(X, \mathcal{A}, \mu)$.
(a) Show that an extended real-valued integrable function is finite a.e. on X.
(b) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions defined on X such that $\sum_{n\in\mathbb{N}} \int_X |f_n| d\mu < \infty$, then show that $\sum_{n\in\mathbb{N}} f_n$ converges a.e. to an integrable function f and

$$\int_X \sum_{n\in\mathbb{N}} f_n d\mu = \int_X f d\mu = \sum_{n\in\mathbb{N}} \int_X f_n d\mu.$$

Solution

(a) Let $E = \{X : |f| = \infty\}$. We want to show that $\mu(E) = 0$. Assume that $\mu(E) > 0$. Since f is integrable

$$\infty > \int_X |f| d\mu \ge \int_E |f| d\mu = \infty.$$

This is a contradiction. Thus, $\mu(E) = 0$.

(b) First we note that $\sum_{n=1}^N |f_n|$ is measurable since f_n is measurable for $n \in \mathbb{N}$. Hence,

$$\lim_{N\to\infty} \sum_{n=1}^N |f_n| = \sum_{n=1}^\infty |f_n|$$

is measurable. Recall that (for nonnegative measurable functions)

$$\int_X \sum_{n=1}^{\infty} |f_n| d\mu = \sum_{n=1}^{\infty} \int_X |f_n| d\mu.$$

By assumption,

$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty,$$

hence,

$$\int_X \sum_{n=1}^{\infty} |f_n| d\mu < \infty.$$

Since $\sum_{n=1}^{\infty} |f_n|$ is integrable on X, by part (a), it is finite a.e. on X. Define a function f as follows:

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} f_n & \text{where } \sum_{n=1}^{\infty} |f_n| < \infty \\ 0 & \text{otherwise.} \end{cases}$$

So f is everywhere defined and $f = \lim_{N\to\infty} \sum_{n=1}^{N} f_n$ a.e. Hence, f is measurable on X. Moreover,

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu = \int_X \left| \sum_{n=1}^{\infty} f_n \right| d\mu \leq \int_X \sum_{n=1}^{\infty} |f_n| d\mu < \infty.$$

Thus, f is integrable and $h_N = \sum_{n=1}^{N} f_n$ converges to f a.e. and

$$|h_N| \leq \sum_{n=1}^{N} |f_n| \leq \sum_{n=1}^{\infty} |f_n|$$

which is integrable. By the D.C.T. we have

$$\begin{aligned} \int_X f d\mu = \int_X \lim_{N\to\infty} h_N d\mu &= \lim_{N\to\infty} \int_X h_N \\ &= \lim_{N\to\infty} \int_X \sum_{n=1}^{N} f_n d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n d\mu \\ &= \sum_{n=1}^{\infty} \int_X f_n d\mu. \quad \blacksquare \end{aligned}$$

Problem 74

Let f be a real-valued Lebesgue measurable function on $[0,\infty)$ such that

1. *f is Lebesgue integrable on every finite subinterval of $[0,\infty)$.*
2. *$\lim_{x\to\infty} f(x) = c \in \mathbb{R}$.*

Show that

$$\lim_{a\to\infty} \frac{1}{a}\int_{[0,a]} f d\mu_L = c.$$

Solution

By assumption 2. we can write

$$(*) \quad \forall \varepsilon > 0, \exists N: \; x > N \;\Rightarrow\; |f(x) - c| < \varepsilon.$$

Now, for $a > N$ we have

$$\begin{aligned}
\left|\frac{1}{a}\int_{[0,a]} f d\mu_L - c\right| &= \left|\frac{1}{a}\int_{[0,a]} (f-c) d\mu_L\right| \\
&\leq \frac{1}{a}\int_{[0,a]} |f-c| d\mu_L \\
&= \frac{1}{a}\left(\int_{[0,N]} |f-c| d\mu_L + \int_{[N,a]} |f-c| d\mu_L\right).
\end{aligned}$$

By (*) we have

$$x \in [N,a] \;\Rightarrow\; |f(x) - c| < \varepsilon.$$

Therefore,

$$(**) \quad \left|\frac{1}{a}\int_{[0,a]} f d\mu_L - c\right| \leq \frac{1}{a}\int_{[0,N]} |f-c| d\mu_L + \frac{(a-N)}{a}\,\varepsilon.$$

It is evident that

$$\lim_{a\to\infty} \frac{(a-N)}{a}\,\varepsilon = \varepsilon.$$

By assumption 1., $|f-c|$ is integrable on $[0,N]$, so $\int_{[0,N]} |f-c| d\mu_L$ is finite and does not depend on a. Hence

$$\lim_{a\to\infty} \frac{1}{a}\int_{[0,N]} |f-c| d\mu_L = 0.$$

Thus, we can rewrite (**) as

$$\lim_{a\to\infty}\left|\frac{1}{a}\int_{[0,a]} f d\mu_L - c\right| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$\lim_{a\to\infty}\left|\frac{1}{a}\int_{[0,a]} f d\mu_L - c\right| = 0. \quad \blacksquare$$

Problem 75
Let f be a non-negative real-valued Lebesgue measurable on $\mathbb{R}$. Show that if $\sum_{n=1}^{\infty} f(x+n)$ is Lebesgue integrable on $\mathbb{R}$, then $f = 0$ a.e. on $\mathbb{R}$.

Solution
Recall these two facts:

1. If $f_n \geq 0$ is measurable on D then $\int_D \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int_D f_n d\mu$.
2. If f is defined and measurable on $\mathbb{R}$ then $\int_{\mathbb{R}} f(x+h) d\mu = \int_{\mathbb{R}} f(x) d\mu$.

From these two facts we have

$$\begin{aligned}\int_{\mathbb{R}}\left(\sum_{n=1}^{\infty} f(x+n)\right) d\mu_L &= \sum_{n=1}^{\infty}\int_{\mathbb{R}} f(x+n) d\mu_L \\ &= \sum_{n=1}^{\infty}\int_{\mathbb{R}} f(x) d\mu_L.\end{aligned}$$

Since $\sum_{n=1}^{\infty} f(x+n)$ is Lebesgue integrable on $\mathbb{R}$,

$$\int_{\mathbb{R}}\left(\sum_{n=1}^{\infty} f(x+n)\right) d\mu_L < \infty.$$

Therefore,

$$(*) \quad \sum_{n=1}^{\infty}\int_{\mathbb{R}} f(x) d\mu_L < \infty.$$

Since $\int_{\mathbb{R}} f(x) d\mu_L \geq 0$, (*) implies that $\int_{\mathbb{R}} f(x) d\mu_L = 0$. Thus, $f = 0$ a.e. on $\mathbb{R}$. $\blacksquare$

Problem 76
Show that the Lebesgue Dominated Convergence Theorem holds if a.e. convergence is replaced by convergence in measure.

Solution
We state the theorem:
Given a measure space $(X, \mathcal{A}, \mu)$. Let $(f_n : n \in \mathbb{N})$ be a sequence of extended real-valued $\mathcal{A}$-measurable functions on $D \in \mathcal{A}$ such that $|f_n| \leq g$ on D for every $n \in \mathbb{N}$ for some integrable non-negative extended real-valued $\mathcal{A}$-measurable function g on D. If $f_n \xrightarrow{\mu} f$ on D, then f is integrable on D and

$$\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu.$$

Proof:
Let (f_{n_k}) be any subsequence of (f_n). Then $f_{n_k} \xrightarrow{\mu} f$ since $f_n \xrightarrow{\mu} f$. By Riesz theorem, there exists a subsequence $(f_{n_{k_l}})$ of (f_{n_k}) such that $f_{n_{k_l}} \to f$ a.e. on D. And we have also $|f_{n_{k_l}}| \leq g$ on D. By the Lebesgue D.C.T. we have

$$(*) \qquad \int_D f d\mu = \lim_{l\to\infty} \int_D f_{n_{k_l}} d\mu.$$

Let $a_n = \int_D f_n d\mu$ and $a = \int_D f d\mu$. Then (*) can be written as

$$\lim_{l\to\infty} a_{n_{k_l}} = a.$$

Hence we can say that any subsequence (a_{n_k}) of (a_n) has a subsequence $(a_{n_{k_l}})$ converging to a. Thus, the original sequence, namely (a_n), converges to the same limit (See Problem 51): $\lim_{n\to\infty} a_n = a$. That is,

$$\lim_{n\to\infty} \int_D f_n d\mu = \int_D f d\mu. \quad \blacksquare$$

Problem 77
Given a measure space $(X, \mathcal{A}, \mu)$. Let $(f_n)_{n\in\mathbb{N}}$ and f be extended real-valued measurable and integrable functions on $D \in \mathcal{A}$.
Suppose that $\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0$. Show that
(a) $f_n \xrightarrow{\mu} f$ on D.
(b) $\lim_{n\to\infty} \int_D |f_n| d\mu = \int_D |f| d\mu$.

Solution

(a) Given any $\varepsilon > 0$, for each $n \in \mathbb{N}$, let $E_n = \{D : |f_n - f| \geq \varepsilon\}$. Then

$$\int_D |f_n - f| d\mu \geq \int_{E_n} |f_n - f| d\mu \geq \varepsilon\mu(E_n).$$

Since $\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0$, $\lim_{n\to\infty} \mu(E_n) = 0$. That is $f_n \xrightarrow{\mu} f$ on D.

(b) Since f_n and f are integrable

$$\int_D (|f_n| - |f|) d\mu = \int_D |f_n| d\mu - \int_D |f| d\mu \leq \int_D |f_n - f| d\mu.$$

By this and the assumption, we get

$$\lim_{n\to\infty} \left(\int_D |f_n| d\mu - \int_D |f| d\mu \right) \leq \lim_{n\to\infty} \int_D |f_n - f| d\mu = 0.$$

This implies

$$\lim_{n\to\infty} \int_D |f_n| d\mu = \int_D |f| d\mu. \quad \blacksquare$$

Problem 78

Given a measure space $(X, \mathcal{A}, \mu)$. Let $(f_n)_{n\in\mathbb{N}}$ and f be extended real-valued measurable and integrable functions on $D \in \mathcal{A}$. Assume that $f_n \to f$ a.e. on D and $\lim_{n\to\infty} \int_D |f_n| d\mu = \int_D |f| d\mu$. Show that

$$\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0.$$

Solution

For each $n \in \mathbb{N}$, let $h_n = |f_n| + |f| - |f_n - f|$. Then $h_n \geq 0$ for all $n \in \mathbb{N}$.
Since $f_n \to f$ a.e. on D, $h_n \to 2|f|$ a.e on D. By Fatou's lemma,

$$\begin{aligned} 2\int_D |f| d\mu &\leq \liminf_{n\to\infty} \int_D (|f_n| + |f|) d\mu - \limsup_{n\to\infty} \int_D |f_n - f| d\mu \\ &= \lim_{n\to\infty} \int_D |f_n| d\mu + \lim_{n\to\infty} \int_D |f| d\mu - \limsup_{n\to\infty} \int_D |f_n - f| d\mu \\ &= 2\int_D |f| d\mu - \limsup_{n\to\infty} \int_D |f_n - f| d\mu. \end{aligned}$$

Since $|f|$ is integrable, we have

$$\limsup_{n\to\infty} \int_D |f_n - f| d\mu \leq 0. \quad (i)$$

Now for each $n \in \mathbb{N}$, let $g_n = |f_n - f| - (|f_n| - |f|)$. Then $h_n \geq 0$ for all $n \in \mathbb{N}$. Since $f_n \to f$ a.e. on D, $g_n \to 0$ a.e on D. By Fatou's lemma,

$$\begin{aligned} 0 = \lim_{n\to\infty} \int_D g_n d\mu &\leq \liminf_{n\to\infty} \int_D |f_n - f| d\mu - \limsup_{n\to\infty} \int_D (|f_n| - |f|) d\mu \\ &\leq \liminf_{n\to\infty} \int_D |f_n - f| d\mu \underbrace{- \lim_{n\to\infty} \int_D |f_n| d\mu + \lim_{n\to\infty} \int_D |f| d\mu)}_{=0}. \end{aligned}$$

Hence

$$\liminf_{n\to\infty} \int_D |f_n - f| d\mu \geq 0. \quad (ii)$$

From (i) and (ii) it follows that

$$\lim_{n\to\infty} \int_D |f_n - f| d\mu = 0. \qquad \blacksquare$$

Problem 79

Let $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ be the Lebesgue space. Let f be an extended real-valued Lebesgue measurable function on $\mathbb{R}$. Show that if f is integrable on $\mathbb{R}$ then

$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| dx = 0.$$

Solution

Since f is integrable,

$$\lim_{M\to\infty} \left(\int_{-\infty}^{-M} |f| dx + \int_M^{\infty} |f| dx \right) = 0 \text{ for } M \in \mathbb{R}.$$

Given any $\varepsilon > 0$, we can pick an $M > 0$ such that

$$\int_{-\infty}^{-M} |f| dx + \int_M^{\infty} |f| dx < \frac{\varepsilon}{4}.$$

Since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, we can find a continuous function φ vanishing outside $[-M, M]$ such that

$$\int_{-M}^{M} |f-\varphi|dx < \frac{\varepsilon}{4}.$$

Then we have

$$\begin{aligned}
\|f-\varphi\|_1 &:= \int_{\mathbb{R}} |f-\varphi|dx \\
&= \int_{-M}^{M} |f-\varphi|dx + \int_{-\infty}^{-M} |f|dx + \int_{M}^{\infty} |f|dx \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

(Recall: $\varphi = 0$ outside $[-M, M]$). Now for any $h \in \mathbb{R}$ we have

$$\|f(x+h)-f(x)\|_1 \leq \|f(x)-\varphi(x)\|_1 + \|\varphi(x)-\varphi(x+h)\|_1 + \|\varphi(x+h)-f(x+h)\|_1.$$

Because of $\varphi \in C_c(\mathbb{R})$ and translation invariance, we have

$$\lim_{h\to 0} \|\varphi(x)-\varphi(x+h)\|_1 = 0 \quad \text{and} \quad \|\varphi(x+h)-f(x+h)\|_1 = \|f(x)-\varphi(x)\|_1.$$

It follows that

$$\begin{aligned}
\lim_{h\to 0} \|f(x+h)-f(x)\|_1 &\leq \|f-\varphi\|_1 + \lim_{h\to 0} \|\varphi(x)-\varphi(x+h)\|_1 + \|f-\varphi\|_1 \\
&\leq 2\,\frac{\varepsilon}{2} + 0 = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{h\to 0} \|f(x+h)-f(x)\|_1 = \lim_{h\to 0} \int_{\mathbb{R}} |f(x+h)-f(x)|dx = 0. \quad \blacksquare$$

Chapter 8

Signed Measures and Radon-Nikodym Theorem

1. Signed measure

Definition 21 *(Signed measure)*
A signed measure on a measurable space $(X, \mathcal{A})$ *is a function* $\lambda : \mathcal{A} \to [-\infty, \infty]$ *such that:*
(1) $\lambda(\varnothing) = 0$.
(2) λ *assumes at most one of the values* $\pm\infty$.
(3) λ *is countably additive. That is, if* $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ *is disjoint, then*

$$\lambda\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \lambda(E_n).$$

Definition 22 *(Positive, negative, null sets)*
Let $(X, \mathcal{A}, \lambda)$ *be a signed measure space. A set* $E \in \mathcal{A}$ *is said to be positive (negative, null) for the signed measure* λ *if*

$$F \in \mathcal{A},\ F \subset E \Longrightarrow \lambda(F) \geq 0\ (\leq 0,\ = 0).$$

Proposition 21 *(Continuity)*
Let $(X, \mathcal{A}, \lambda)$ *be a signed measure space.*
1. If $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ *is an increasing sequence then*

$$\lim_{n\to\infty} \lambda(E_n) = \lim_{n\to\infty} \lambda\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lambda\Big(\lim_{n\to\infty} E_n\Big).$$

2. If $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ *is an decreasing sequence and* $\lambda(E_1) < \infty$*, then*

$$\lim_{n\to\infty} \lambda(E_n) = \lim_{n\to\infty} \lambda\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \lambda\Big(\lim_{n\to\infty} E_n\Big).$$

Proposition 22 *(Some more properties)*
Let $(X, \mathcal{A}, \lambda)$ be a signed measure space.
1. Every measurable subset of a positive (negative, null) set is a positive (negative, null) set.
2. If E is a positive set and F is a negative set, then $E \cap F$ is a null set.
3. Union of positive (negative, null) sets is a positive (negative, null) set.

Theorem 10 *(Hahn decomposition theorem)*
Let $(X, \mathcal{A}, \lambda)$ be a signed measure space. Then there is a positive set A and a negative set B such that

$$A \cap B = \varnothing \quad \text{and} \quad A \cup B = X.$$

Moreover, if A' and B' are another pair, then $A \triangle A'$ and $B \triangle B'$ are null sets.
$\{A, B\}$ is called a Hahn decomposition of $(X, \mathcal{A}, \lambda)$.

Definition 23 *(Singularity)*
Two signed measure λ_1 and λ_2 on a measurable space $(X, \mathcal{A})$ are said to be mutually singular and we write $\lambda_1 \perp \lambda_2$ if there exist two set $E, F \in \mathcal{A}$ such that $E \cap F = \varnothing,\ E \cup F = X,\ E$ is a null set for λ_1 and F is a null set for λ_2.

Definition 24 *(Jordan decomposition)*
Given a signed measure space $(X, \mathcal{A}, \lambda)$. If there exist two positive measures μ and ν, at least one of which is finite, on the measurable $(X, \mathcal{A})$ such that

$$\mu \perp \nu \quad \text{and} \quad \lambda = \mu - \nu,$$

then $\{\mu, \nu\}$ is called a Jordan decomposition of λ.

Theorem 11 *(Jordan decomposition of signed measures)*
Given a signed measure space $(X, \mathcal{A}, \lambda)$. A Jordan decomposition for $(X, \mathcal{A}, \lambda)$ exists and unique, that is, there exist a unique pair $\{\mu, \nu\}$ of positive measures on $(X, \mathcal{A})$, at least one of which is finite, such that $\mu \perp \nu$ and $\lambda = \mu - \nu$.
Moreover, with any arbitrary Hahn decomposition $\{A, B\}$ of $(X, \mathcal{A}, \lambda)$, if we define two set functions μ and ν by setting

$$\mu(E) = \lambda(E \cap A) \quad \text{and} \quad \nu(E) = -\lambda(E \cap B) \quad \text{for} \quad E \in \mathcal{A},$$

then $\{\mu, \nu\}$ is a Jordan decomposition for $(X, \mathcal{A}, \lambda)$.

2. Lebesgue decomposition, Radon-Nikodym Theorm

Definition 25 *(Radon-Nikodym derivative)*
Let μ be a positive measure and λ be a signed measure on a measurable space $(X, \mathcal{A})$. If there exists an extended real-valued $\mathcal{A}$-measurable function f on X such that

$$\lambda(E) = \int_E f d\mu \quad \text{for every} \quad E \in \mathcal{A},$$

then f is called a Radon-Nikodym derivative of λ with respect to μ, and we write $\frac{d\lambda}{d\mu}$ for it.

Proposition 23 *(Uniqueness)*
Let μ be a σ-finite positive measure and λ be a signed measure on a measurable space $(X, \mathcal{A})$. If two extended real-valued $\mathcal{A}$-measurable functions f and g are Radon-Nikodym derivatives of λ with respect to μ, then $f = g$ μ-a.e. on X.

Definition 26 *(Absolute continuity)*
Let μ be a positive measure and λ be a signed measure on a measurable space $(X, \mathcal{A})$. We say that λ is absolutely continuous with respect to μ and write $\lambda \ll \mu$ if

$$\forall E \in \mathcal{A}), \ \mu(E) = 0 \Longrightarrow \lambda(E) = 0.$$

Definition 27 *(Lebesgue decomposition)*
Let μ be a positive measure and λ be a signed measure on a measurable space $(X, \mathcal{A})$. If there exist two signed measures λ_a and λ_s on $(X, \mathcal{A})$ such that

$$\lambda_a \ll \mu, \quad \lambda_s \perp \mu \quad \textit{and} \quad \lambda = \lambda_a + \lambda_s,$$

then we call $\{\lambda_a, \lambda_s\}$ a Lebesgue decomposition of λ with respect to μ. We call λ_a and λ_s the absolutely continuous part and the singular part of λ with respect to μ.

Theorem 12 *(Existence of Lebesgue decomposition)*
Let μ be a σ-finite positive measure and λ be a σ-finite signed measure on a measurable space $(X, \mathcal{A})$. Then there exist two signed measures λ_a and λ_s on $(X, \mathcal{A})$ such that

$$\lambda_a \ll \mu, \ \lambda_s \perp \mu, \ \lambda = \lambda_a + \lambda_s \quad \textit{and } \lambda_a \textit{ is defined by} \ \ \lambda_a(E) = \int_E f d\mu, \ \forall E \in \mathcal{A},$$

where f is an extended real-valued measurable function on X.

Theorem 13 *(Radon-Nikodym theorem)*
Let μ be a σ-finite positive measure and λ be a σ-finite signed measure on a measurable space $(X, \mathcal{A})$. If $\lambda \ll \mu$, then the Radon-Nikodym derivative of λ with respect to μ exists, that is, there exists an extended real-valued measurable function on X such that

$$\lambda(E) = \int_E f d\mu, \ \forall E \in \mathcal{A}.$$

* * **

Problem 80
Given a signed measure space $(X, \mathcal{A}, \lambda)$. Suppose that $\{\mu, \nu\}$ is a Jordan decomposition of λ, and E and F are two measurable subsets of X such that $E \cap F = \varnothing$, $E \cup F = X$, E is a null set for ν and F is a null set for ν.Show that $\{E, F\}$ is a Hahn decomposition for $(X, \mathcal{A}, \lambda)$.

Solution
We show that E is a positive set for λ and F is a negative set for λ. Since $\{\mu, \nu\}$ is a Jordan decomposition of λ, we have

$$\lambda(E) = \mu(E) - \nu(E),\ \forall E \in \mathcal{A}.$$

Let $E_0 \in \mathcal{A},\ E_0 \subset E$. Since E is a null set for ν, E_0 is also a null set for ν. Thus $\nu(E_0) = 0$. Consequently, $\lambda(E_0) = \mu(E_0) \geq 0$. This shows that E is a positive set for λ.
Similarly, let $F_0 \in \mathcal{A},\ F_0 \subset E$. Since F is a null set for μ, F_0 is also a null set for μ. Thus $\mu(F_0) = 0$. Consequently, $\lambda(F_0) = -\nu(F_0) \leq 0$. This shows that F is a negative set for λ.
We conclude that $\{E, F\}$ is a Hahn decomposition for $(X, \mathcal{A}, \lambda)$. ■

Problem 81
Consider a measure space $([0, 2\pi], \mathcal{M}_L \cap [0, 2\pi], \mu_L)$. *Define a signed measure* λ *on this space by setting*

$$\lambda(E) = \int_E \sin x d\mu_L, \quad \textit{for} \quad E \in \mathfrak{M}_L \cap [0, 2\pi].$$

Let $C = [\frac{4}{3}\pi, \frac{5}{3}\pi]$. *Let* $\varepsilon > 0$ *be arbitrary given. Find a measurable set* $C' \subset C$ *such that* $\lambda(C') \geq \lambda(C)$ *and* $\lambda(E) > -\varepsilon$ *for every measurable subset* E *of* C'.

Solution
Let $X = [0, 2\pi],\ f(x) = \sin x$. Then f is continuous on X, so f is Lebesgue (=Riemann) integrable on X. Given $\varepsilon > 0$, let $\delta = \min\{\frac{\varepsilon}{2}, \frac{\pi}{3}\}$. Let $C' = [\frac{4}{3}\pi, \frac{4}{3}\pi + \delta]$, then

$$C' \subset C \quad \text{and} \quad f(x) = \sin x < 0,\ x \in C'.$$

We have

$$\lambda(C') = \int_{C'} \sin x d\mu_L \geq \int_C \sin x d\mu_L = \lambda(C).$$

Now for any $E \subset C'$ and $E \in \mathcal{M}_L \cap [0, 2\pi]$, since $\mu(E) \leq \mu(C')$ and $f(x) \leq 0$ on C', we have

$$\lambda(E) = \int_E \sin x d\mu_L \geq \int_{C'} \sin x d\mu_L \geq \int_{C'} (-1) d\mu_L = -\mu(C') = -\delta.$$

By the choice of δ, we have

$$\delta < \frac{\varepsilon}{2} \ \Rightarrow\ -\delta > -\frac{\varepsilon}{2} > -\varepsilon.$$

Thus, for any $E \in \mathcal{M}_L \cap [0, 2\pi]$ with $E \subset C'$ we have $\lambda(E) > -\varepsilon$. ■

Problem 82
Given a signed measure space $(X, \mathcal{A}, \lambda)$.
(a) Show that if $E \in \mathcal{A}$ and $\lambda(E) > 0$, then there exists a subset $E_0 \subset E$ which is a positive set for λ with $\lambda(E_0) \geq \lambda(E)$.
(b) Show that if $E \in \mathcal{A}$ and $\lambda(E) < 0$, then there exists a subset $E_0 \subset E$ which is a negative set for λ with $\lambda(E_0) \leq \lambda(E)$.

Solution
(a) If E is a positive set for λ then we're done (just take $E_0 = E$).
Suppose E is a not positive set for λ. Let $\{A, B\}$ be a Hahn decomposition of $(X, \mathcal{A}, \lambda)$. Let $E_0 = E \cap A$. Since A is a positive set, so E_0 is also a positive set (for $E_0 \subset A$). Moreover,

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \cap B) = \lambda(E_0) + \lambda(E \cap B).$$

Since $\lambda(E \cap B) \leq 0$, $0 < \lambda(E) \leq \lambda(E_0)$. Thus, $E_0 = E \cap A$ is the desired set.

(b) Similar argument. Answer: $E_0 = E \cap B$. ■

Problem 83
Let μ and ν two positive measures on a measurable space $(X, \mathcal{A})$. Suppose for every $\varepsilon > 0$, there exists $E \in \mathcal{A}$ such that $\mu(E) < \varepsilon$ and $\nu(E^c) < \varepsilon$. Show that $\mu \perp \nu$.

Solution
Recall: For positive measures μ and ν

$$\mu \perp \nu \;\Leftrightarrow\; \exists A \in \mathcal{A}: \; \mu(A) = 0 \text{ and } \nu(A^c) = 0.$$

By hypothesis, for every $n \in \mathbb{N}$, there exists $E_n \in \mathcal{A}$ such that

$$\mu(E_n) < \frac{1}{n^2} \text{ and } \nu(E_n^c) < \frac{1}{n^2}.$$

Hence,

$$\sum_{n\in\mathbb{N}} \mu(E_n) \leq \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty \text{ and } \sum_{n\in\mathbb{N}} \nu(E_n^c) \leq \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty.$$

By Borel-Cantelli's lemma we get

$$\mu\left(\limsup_{n\to\infty} E_n\right) = 0 \quad \text{and} \quad \nu\left(\limsup_{n\to\infty} E_n^c\right) = 0.$$

Let $A = \limsup_{n\to\infty} E_n$. Then $\mu(A) = 0$. (*)
We claim: $A^c = \liminf_{n\to\infty} E_n^c$. Recall:

$$\liminf_{n\to\infty} A_n = \{x \in X : \ x \in A_n \quad \text{for all but finitely many} \ \ n \in \mathbb{N}\}.$$

For every $x \in X$, for each $n \in \mathbb{N}$, we have either $x \in E_n$ or $x \in E_n^c$. If $x \in E_n$ for infinitely many n, then $x \in \limsup_{n\to\infty} E_n$ and vice versa. Otherwise, $x \in E_n$ for a finite numbers of n. But this is equivalent to $x \in E_n^c$ for all but finitely many n. That is $x \in \liminf_{n\to\infty} E_n^c$. Hence,

$$\limsup_{n\to\infty} E_n \cup \liminf_{n\to\infty} E_n^c = X.$$

Now, if $x \in \limsup_{n\to\infty} E_n$ then $x \in E_n$ for infinitely many n, so $x \notin \liminf_{n\to\infty} E_n^c$. This shows that

$$\limsup_{n\to\infty} E_n \cap \liminf_{n\to\infty} E_n^c = \emptyset.$$

Thus, $A^c = \liminf_{n\to\infty} E_n^c$ as required.
Last, we show that $\nu(A^c) = 0$. Since $\liminf_{n\to\infty} E_n^c \subset \limsup_{n\to\infty} E_n^c$ and $\nu\left(\limsup_{n\to\infty} E_n^c\right) = 0$ (by the first paragraph), we get

$$\nu(\liminf_{n\to\infty} E_n^c) = \nu(A^c) = 0. \qquad (**)$$

From (*) and (**) we obtain $\mu\perp\nu$. ■

Problem 84
Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Let ν be the counting measure on $\mathcal{M}_L$, that is, ν is defined by setting $\nu(E)$ to be equal to the numbers of elements in $E \in \mathcal{M}_L$ if E is a finite set and $\nu(E) = \infty$ if E is infinite set.
(a) Show that $\mu_L \ll \nu$ but $\frac{d\mu_L}{d\nu}$ does not exist.
(b) Show that ν does not have a Lebesgue decomposition with respect to μ_L.

Solution
(a) Let $E \subset \mathbb{R}$ with $\nu(E) = 0$. Since ν be the counting measure, $E = \emptyset$. Then $\mu_L(E) = \mu_L(\emptyset) = 0$. Thus,

$$E \subset \mathbb{R}, \ \nu(E) = 0 \ \Rightarrow \ \mu_L(E) = 0.$$

Hence, $\mu_L \ll \nu$.
Suppose there exists a measurable function f such that

$$m_L(E) = \int_E f d\nu \quad \text{for every} \quad E \in \mathcal{M}_L.$$

Take $E = \{x\},\ x \in \mathbb{R}$ then we have

$$E \in \mathcal{M}_L,\ \mu_L(E) = 0, \quad \text{and} \quad \nu(E) = 1.$$

This implies that $f \equiv 0$. Then for every $A \in \mathcal{M}_L$ we have

$$\mu_L(A) = \int_A 0 d\nu = 0.$$

This is impossible.
(b) Assume that ν have a Lebesgue decomposition with respect to μ_L. Then, for every $E \subset \mathbb{R}$ and some measurable function f,

$$\nu = \nu_a + \nu_s,\ \nu_a \ll \mu_L,\ \nu_s \perp \mu_L, \quad \text{and} \quad \nu_a(E) = \int_E f d\mu_L.$$

Since $\nu_s \perp \mu_L$, there exists $A \in \mathcal{M}_L$ such that $\mu_L(A^c) = 0$ and A is a null set for ν_s. Pick $a \in A$ then $\nu_s(\{a\}) = 0$. On the other hand,

$$\nu_a(\{a\}) = \int_{\{a\}} f d\mu_L \quad \text{and} \quad \mu_L(\{a\}) = 0.$$

It follows that $\nu_a(\{a\}) = 0$. Since $\nu = \nu_a + \nu_s$, we get

$$1 = \nu(\{a\}) = \nu_a(\{a\}) + \nu_s(\{a\}) = 0 + 0 = 0.$$

This is a contradiction. Thus, ν does not have a Lebesgue decomposition with respect to μ_L. ■

Problem 85
Let μ and ν be two positive measures on a measurable space $(X, \mathcal{A})$.
(a) Show that if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\nu(E) < \varepsilon$ for every $E \in \mathcal{A}$ with $\mu(E) < \delta$, then $\nu \ll \mu$.
(b) Show that if ν is a finite positive measure, then the converse of (a) holds.

Solution
(a) Suppose this statement is true: *(*):= for every $\varepsilon > 0$ there exists $\delta > 0$ such*

that $\nu(E) < \varepsilon$ for every $E \in \mathcal{A}$ with $\mu(E) < \delta$.
Take $E \in \mathcal{A}$ with $\mu(E) = 0$. Then

$$\forall \varepsilon > 0, \ \nu(E) < \varepsilon.$$

It follows that $\nu(E) = 0$. Hence $\nu \ll \mu$.
(b) Suppose ν is a finite positive measure and μ is a positive measure such that $\nu \ll \mu$. We want to show (*) is true. Assume that (*) is false. that is

$$\exists \varepsilon > 0 \ \text{ st } \ \big[\forall \delta > 0, \ \exists E \in \mathcal{A} \ \text{ st } \ \{\mu(E) < \delta \ \text{ and } \ \nu(E) \geq \varepsilon\}\big].$$

In particular,

$$\exists \varepsilon > 0 \ \text{ st } \ \left[\forall n \in \mathbb{N}, \ \exists E_n \in \mathcal{A} \ \text{ st } \ \{\mu(E_n) < \frac{1}{n^2} \ \text{ and } \ \nu(E_n) \geq \varepsilon\}\right]$$

Since $\sum_{n\in\mathbb{N}} \mu(E_n) \leq \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$, by Borel-Catelli lemma, we have

$$\mu(\limsup_{n\to\infty} E_n) = 0.$$

Set $E = \limsup_{n\to\infty} E_n$, then $\mu(E) = 0$. Since $\nu \ll \mu$, $\nu(E) = 0$. Note that $\nu(X) < \infty$, we have

$$\nu(E) = \nu(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} \nu(E_n) \geq \nu(E_n) \geq \varepsilon.$$

This is a contradiction. Thus, (*) must be true. ∎

Problem 86
Let μ and ν be two positive measures on a measurable space $(X, \mathcal{A})$. Suppose $\frac{d\nu}{d\mu}$ exists so that $\nu \ll \mu$.
(a) Show that if $\frac{d\nu}{d\mu} > 0$, μ-a.e. on X, then $\mu \ll \nu$ and thus, $\mu \sim \nu$.
(b) Show that if $\frac{d\nu}{d\mu} > 0$, μ-a.e. on X and if μ and ν are σ-finite, then $\frac{d\mu}{d\nu}$ exists and

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1}, \quad \mu - a.e. \text{ and } \nu - a.e. \text{ on } X.$$

Solution
(a) For every $E \in \mathcal{A}$, by definition, we have

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu.$$

Suppose $\nu(E) = 0$. Since $\frac{d\nu}{d\mu} > 0$, μ-a.e. on X, we have

$$\int_E \frac{d\nu}{d\mu} d\mu = 0.$$

Hence, $\mu(E) = 0$. This implies that $\mu \ll \nu$ and so $\mu \sim \nu$ (since $\nu \ll \mu$ is given).

(b) Suppose $\frac{d\nu}{d\mu} > 0$, μ-a.e. on X and if μ and ν are σ-finite. The existence of $\frac{d\mu}{d\nu}$ is guaranteed by the Radon-Nikodym theorem (since $\mu \sim \nu$ by part a). Moreover,

$$\frac{d\mu}{d\nu} > 0, \quad \nu - a.e. \text{ on } \ X.$$

By the chain rule,

$$\begin{aligned} \frac{d\mu}{d\nu} . \frac{d\nu}{d\mu} &= \frac{d\mu}{d\mu} = 1, \quad \mu - a.e. \text{ on } \ X. \\ \frac{d\nu}{d\mu} . \frac{d\mu}{d\nu} &= \frac{d\nu}{d\nu} = 1, \quad \nu - a.e. \text{ on } \ X. \end{aligned}$$

Thus,

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu} \right)^{-1}, \quad \mu - a.e. \text{ and } \nu - a.e. \text{ on } \ X. \quad \blacksquare$$

Problem 87

Let $(X, \mathcal{A}, \mu)$ be a measure space. Assume that there exists a measurable function $f : X \to (0, \infty)$ satisfying the condition that $\mu\{x \in X : \ f(x) \leq n\} < \infty$ for every $n \in \mathbb{N}$.

(a) Show that the existence of such a function f implies that μ is a σ-finite measure.

(b) Define a positive measure ν on $\mathcal{A}$ by setting

$$\nu(E) = \int_E f d\mu \quad \textit{for} \quad E \in \mathcal{A}.$$

Show that ν is a σ-finite measure.

(c) Show that $\frac{d\mu}{d\nu}$ exists and

$$\frac{d\mu}{d\nu} = \frac{1}{f}, \quad \mu - a.e. \textit{ and } \nu - a.e. \textit{ on } X.$$

Solution
(a)By assumption, $\mu\{x \in X : f(x) \leq n\} < \infty$ for every $n \in \mathbb{N}$. Since $0 < f < \infty$, so $\bigcup_{n=1}^{\infty}\{X : f \leq n\} = X$. Hence μ is a σ-finite measure.

(b) Let $\nu(E) = \int_E f d\mu$ for $E \in \mathcal{A}$.
Since $f > 0$, ν is a positive measure and if $\mu(E) = 0$ then $\nu(E) = 0$. Hence $\nu \ll \mu$. Conversely, if $\nu(E) = 0$, since $f > 0$, $\mu(E) = 0$. So $\mu \ll \nu$. Thus, $\mu \sim \nu$. Since μ is σ-finite (by (a)), ν is also σ-finite.

(c) Since ν is σ-finite, $\frac{d\mu}{d\nu}$ exists. By part (b), $f = \frac{d\nu}{d\mu}$. By chain rule,

$$\frac{d\mu}{d\nu}.\frac{d\nu}{d\mu} = 1, \quad \mu - a.e. \text{ and } \nu - a.e. \text{ on } X.$$

Thus,

$$\frac{d\mu}{d\nu} = \frac{1}{f}, \ \mu - a.e. \text{ and } \nu - a.e. \text{ on } X. \quad \blacksquare$$

Problem 88
Let μ and ν be σ-finite positive measures on $(X, \mathcal{A})$. Show that there exist $A, B \in \mathcal{A}$ such that

$$A \cap B = \varnothing, \ A \cap B = X, \ \mu \sim \nu \ \text{ on } \ (A, \mathcal{A} \cap A) \ \text{ and } \ \mu \perp \nu \ \text{ on } \ (B, \mathcal{A} \cap B).$$

Solution
Define a σ-finite measure $\lambda = \mu + \nu$. Then $\mu \ll \lambda$ and $\nu \ll \lambda$. By the Radon-Nikodym theorem there exist non-negative $\mathcal{A}$-measurable functions f and g such that for every $E \in \mathcal{A}$,

$$\mu(E) = \int_E f d\lambda \ \text{ and } \ \nu(E) = \int_E g d\lambda.$$

Let $A = \{x \in X : f(x)g(x) > 0\}$ and $B = A^c$. Then $\mu \sim \nu$. Indeed, $f > 0$ in A. Thus, if $\mu(E) = 0$, then $\lambda(E) = 0$, and therefore, $\nu(E) = 0$. This implies $\nu \ll \mu$. We can prove $\mu \ll \nu$ in the same manner. Hence, $\mu \sim \nu$.
Let $C = \{x \in B : f(x) = 0\}$, $D = B \setminus C$. For any measurable sets $E \subset C$ and $F \subset D$, $\mu(E) = \nu(F) = 0$. Thus, $\mu \perp \nu$ on $(B, \mathcal{A} \cap B)$. $\blacksquare$

Problem 89
Let μ and ν be σ-finite positive measures on $(X, \mathcal{A})$. Show that there exists a non-negative extended real-valued $\mathcal{A}$-measurable function φ on X and a set $A_0 \in \mathcal{A}$ with $\mu(A_0) = 0$ such that

$$\nu(E) = \int_E \varphi d\mu + \nu(E \cap A_0) \quad \textit{for every} \quad E \in \mathcal{A}.$$

Solution
By the Lebesgue decomposition theorem,

$$\nu = \nu_a + \nu_s, \ \nu_a \ll \mu, \ \nu_s \perp \mu \ \text{ and } \quad \nu_a(E) = \int_E \varphi d\mu \ \text{ for any } \ E \in \mathcal{A},$$

where φ is a non-negative extended real-valued $\mathcal{A}$-measurable function on X. Now since $\nu_s \perp \mu$, there exists $A_0 \in \mathcal{A}$ such that

$$\mu(A_0) = 0 \ \text{ and } \ \nu_s(A_0^c) = 0.$$

Hence

$$\big[\nu_a \ll \mu \text{ and } \mu(A_0) = 0\big] \Longrightarrow \nu_a(A_0) = 0. \quad (*)$$

On the other hand, since $\nu_s(E) = \nu_s(E \cap A_0)$ for every $E \in \mathcal{A}$, so we have

$$\nu(E \cap A_0) = \underbrace{\nu_a(E \cap A_0)}_{=0 \text{ by } (*)} + \nu_s(E \cap A_0) = \nu_s(E \cap A_0) = \nu_s(E).$$

Finally,

$$\nu(E) = \nu_a(E) + \nu_s(E) = \int_E \varphi d\mu + \nu(E \cap A_0) \ \text{ for every } \ E \in \mathcal{A}. \quad \blacksquare$$

Chapter 9

Differentiation and Integration

The measure space in this chapter is the space $(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Therefore, we write μ instead of μ_L for the Lebesgue measure. Also, we say f is integrable (derivable) instead of f is μ_L-integrable (derivable).

1. BV functions and absolutely continuous functions

Definition 28 *(Variation of f)*
Let $[a, b] \subset \mathbb{R}$ with $a < b$. A partition of $[a, b]$ is a finite ordered set $\mathcal{P} = \{a = x_0 < x_1 < ... < x_n = b\}$. For a real-valued function f on $[a, b]$ we define the variation of f corresponding to a partition $\mathcal{P}$ by

$$V_a^b(f, \mathcal{P}) := \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \in [0, \infty).$$

We define the total variation of f on $[a, b]$ by

$$V_a^b(f) := \sup_{\mathcal{P}} V_a^b(f, \mathcal{P}) \in [0, \infty],$$

where the supremum is taken over all partitions of $[a, b]$. We say that f is a function of bounded variation on $[a, b]$, or simply a BV function, if $V_a^b(f) < \infty$.
We write $BV([a, b])$ for the collection of all BV functions on $[a, b]$.

Theorem 14 *(Jordan decomposition of a BV function)*
1. A function f is a BV function on $[a, b]$ if and only if there are two real-valued increasing functions g_1 and g_2 on $[a, b]$ such that $f = g_1 - g_2$ on $[a, b]$.
$\{g_1, g_2\}$ is called a Jordan decomposition of f.
2. If a BV function on $[a, b]$ is continuous on $[a, b]$, then g_1 and g_2 can be chosen to be continuous on $[a, b]$.

Theorem 15 *(Derivability and integrability)*
If f is a BV function on $[a, b]$, then f' exists a.e. on $[a, b]$ and integrable on $[a, b]$.

Definition 29 *(Absolutely continuous functions)*
A real-valued function f on $[a,b]$ is said to be absolutely continuous on $[a,b]$ if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every finite collection $\{[a_k, b_k]\}_{1 \leq k \leq n}$ of non-overlapping intervals contained in $[a,b]$ with

$$\sum_{k=1}^{n} |b_k - a_k| < \delta.$$

Theorem 16 *(Properties)*
If f is an absolutely continuous on $[a,b]$ then
1. f is uniformly continuous on $[a,b]$,
2. f is a BV function on $[a,b]$,
3. f' exists a.e. on $[a,b]$,
4. f is integrable on $[a,b]$.

Definition 30 *(Condition (N))*
Let f be a real-valued function on $[a,b]$. We say that f satisfies Lusin's Condition (N) on $[a,b]$ if for every $E \subset [a,b]$ with $\mu_L(E) = 0$, we have $\mu\big(f(E)\big) = 0$.

Theorem 17 *(Banach-Zarecki criterion for absolute continuity)*
Let f be a real-valued function on $[a,b]$. Then f is absolutely continuous on $[a,b]$ if and only if it satisfies the following three conditions:
1. f is continuous on $[a,b]$.
2. f is of BV on $[a,b]$.
3. f satisfies condition (N) on $[a,b]$.

2. Indefinite integrals and absolutely continuous functions

Definition 31 *(Indefinite integrals)*
Let f be a extended real-valued function on $[a,b]$. Suppose that f is measurable and integrable on $[a,b]$. By indefinite integral of f on $[a,b]$ we mean a real-valued function F on $[a,b]$ defined by

$$F(x) = \int_{[a,x]} f d\mu + c, \quad x \in [a,b] \quad \text{and } c \in \mathbb{R} \text{ is a constant.}$$

Theorem 18 *(Lebesgue differentiation theorem)*
Let f be a extended real-valued, measurable and integrable function on $[a,b]$. Let F be an indefinite integral of f on $[a,b]$. Then
1. F is absolutely continuous on $[a,b]$,
2. F' exists a.e. on $[a,b]$ and $F' = f$ a.e. on $[a,b]$,

Theorem 19 *Let f be a real-valued absolutely continuous on $[a, b]$. Then*

$$\int_{[a,x]} f' d\mu = f(x) - f(a), \ \forall x \in [a, b].$$

Thus, an absolutely continuous function is an indefinite integral of its derivative.

Theorem 20 *(A characterization of an absolutely continuous function)*
A real-valued function f on $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it satisfies the following conditions:
(i) f' exists a.e. on $[a, b]$
(ii) f' is measurable and integrable on $[a, b]$.
(iii) $\int_{[a,x]} f' d\mu = f(x) - f(a), \ \forall x \in [a, b]$.

3. Indefinite integrals and BV functions

Theorem 21 *(Total variation of F)*
Let f be a extended real-valued measurable and integrable function on $[a, b]$. Let F be an indefinite integral of f on $[a, b]$ defined by

$$F(x) = \int_{[a,x]} f d\mu + c, \quad x \in [a, b].$$

Then the total variation of F is given by

$$V_a^b(F) = \int_{[a,b]} |f| d\mu.$$

* * **

Problem 90
Let $f \in BV([a, b])$. Show that if $f \geq c$ on $[a, b]$ for some constant $c > 0$, then $\frac{1}{f} \in BV([a, b])$.

Solution
Let $\mathcal{P} = \{a = x_0 < x_1 < ... < x_n = b\}$ be a partition of $[a, b]$. Then

$$V_a^b\Big(\frac{1}{f}, \mathcal{P}\Big) = \sum_{k=1}^{n} \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right| = \sum_{k=1}^{n} \frac{|f(x_k) - f(x_{k-1})|}{|f(x_k) f(x_{k-1})|}.$$

Since $f \geq c > 0$,

$$\frac{|f(x_k) - f(x_{k-1})|}{|f(x_k) f(x_{k-1}|)} \leq \frac{|f(x_k) - f(x_{k-1})|}{c^2}.$$

It follows that

$$V_a^b\Big(\frac{1}{f},\mathcal{P}\Big) \leq \frac{1}{c^2}\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})| = \frac{1}{c^2}V_a^b(f,\mathcal{P}) \leq \frac{1}{c^2}V_a^b(f).$$

Since $V_a^b(f)<\infty,\ V_a^b\big(\frac{1}{f}\big)<\infty.$ ■

Problem 91
Let $f,g\in BV([a,b])$. Show that $fg\in BV([a,b])$ and

$$V_a^b(fg) \leq \sup_{[a,b]}|f|.V_a^b(g) + \sup_{[a,b]}|g|.V_a^b(f).$$

Solution
Note first that $f,g\in BV([a,b])$ implies that f and g are bounded on $[a,b]$. There are some $0<M<\infty$ and $0<N<\infty$ such that

$$M=\sup_{[a,b]}|f| \text{ and } N=\sup_{[a,b]}|g|.$$

For any $x,y\in[a,b]$ we have

$$\begin{aligned}|f(x)g(x)-f(y)g(y)| &\leq |f(x)-f(y)||g(x)|+|g(x)-g(y)||f(y)|\\ &\leq N|f(x)-f(y)|+M|g(x)-g(y)| \quad (*).\end{aligned}$$

Now, let $\mathcal{P}=\{a=x_0<x_1<...<x_n=b\}$ be any partition of $[a,b]$. Then we have

$$\begin{aligned}V_a^b(fg,\mathcal{P}) &= \sum_{k=1}^{n}|f(x_k)g(x_k)-f(x_{k-1})g(x_{k-1})|\\ &\leq M\sum_{k=1}^{n}|g(x_k)-g(x_{k-1})| + N\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\\ &\leq MV_a^b(g,\mathcal{P})+NV_a^b(f,\mathcal{P}).\end{aligned}$$

Since $\mathcal{P}$ is arbitrary,

$$\sup_{\mathcal{P}}V_a^b(fg,\mathcal{P}) \leq M\sup_{\mathcal{P}}V_a^b(g,\mathcal{P})+N\sup_{\mathcal{P}}V_a^b(f,\mathcal{P}),$$

where the supremum is taken over all partitions of $[a,b]$. Thus,

$$V_a^b(fg) \leq \sup_{[a,b]}|f|.V_a^b(g) + \sup_{[a,b]}|g|.V_a^b(f). \quad ■$$

Problem 92
Let f be a real-valued function on $[a,b]$. Suppose f is continuous on $[a,b]$ and satisfying the Lipschitz condition, that is, there exists a constant $M > 0$ such that

$$|f(x') - f(x'')| \leq M|x' - x''|, \ \forall x', x'' \in [a,b].$$

Show that $f \in BV([a,b])$ and $V_a^b(f) \leq M(b-a)$.

Solution
Let $\mathcal{P} = \{a = x_0 < x_1 < ... < x_n = b\}$ be any partition of $[a,b]$. Then

$$\begin{aligned} V_a^b(f, \mathcal{P}) &= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\ &\leq M \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &\leq M(x_n - x_0) = M(b-a). \end{aligned}$$

This implies that

$$V_a^b(f) = \sup_{\mathcal{P}} V_a^b(f, \mathcal{P}) \leq M(b-a) < \infty. \quad \blacksquare$$

Problem 93
Let f be a real-valued function on $[a,b]$. Suppose f is continuous on $[a,b]$ and is differentiable on (a,b) with $|f'| \leq M$ for some constant $M > 0$. Show that $f \in BV([a,b])$ and $V_a^b(f) \leq M(b-a)$.

Hint:
Show that f satisfies the Lipschitz condition.

Problem 94
Let f be a real-valued function on $[0, \frac{2}{\pi}]$ defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{for } x \in (0, \frac{2}{\pi}] \\ 0 & \text{for } x = 0. \end{cases}$$

Show that $f \notin BV([0, \frac{2}{\pi}])$.

Solution
Let us choose a particular partition of $[0, \frac{2}{\pi}]$:

$$x_1 = \frac{2}{\pi} > x_2 = \frac{2}{\pi + 2\pi} > ... > x_{2n-1} = \frac{2}{\pi + 2n.2\pi} > x_{2n} = 0.$$

Then we have

$$\begin{aligned} V_0^{\frac{2}{\pi}}(f, \mathcal{P}) &= |f(x_1) - f(x_2)| + |f(x_2) - f(x_3)| + ... + |f(x_{2n-1}) - f(x_{2n})| \\ &= \underbrace{2 + 2 + ... + 2}_{2n-1} + 1 = (2n-1)2 + 1. \end{aligned}$$

Therefore,

$$\sup_{\mathcal{P}} V_0^{\frac{2}{\pi}}(f, \mathcal{P}) = \infty,$$

where the supremum is taken over all partitions of $[0, \frac{2}{\pi}]$. Thus, f is not a BV function. ■

Problem 95
Let f be a real-valued continuous and BV function on $[0,1]$. Show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^2 = 0.$$

Solution
Since f is continuous on $[0,1]$, which is compact, f is uniformly continuous on $[0,1]$. Hence,

$$\forall \varepsilon > 0,\ \exists N > 0 :\ |x - y| \leq \frac{1}{N} \ \Rightarrow\ |f(x) - f(y)| \leq \varepsilon,\ \forall x, y \in [0,1].$$

Partition of $[0,1]$:

$$x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < ... < x_n = \frac{n}{n} = 1.$$

For $n \geq N$ we have $\left|\frac{i}{n} - \frac{i-1}{n}\right| = \frac{1}{n} \leq \frac{1}{N}$. Hence,

$$\left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| \leq \varepsilon,\ i = 1, 2, ...$$

Now we can write, for $n \geq N$,

$$\begin{aligned}\left|f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right|^2 &= \left|f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right|.\left|f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right| \\ &\leq \varepsilon\left|f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right|,\end{aligned}$$

and so

$$\sum_{i=1}^{n}\left|f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right|^2 \leq \varepsilon\sum_{i=1}^{n}\left|f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right| \leq \varepsilon V_0^1(f).$$

Since $V_0^1(f) < \infty$, we can conclude that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left|f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right|^2 = 0. \quad \blacksquare$$

Problem 96
Let $(f_i :\ i \in \mathbb{N})$ *and* f *be real-valued functions on an interval* $[a,b]$ *such that* $\lim_{i\to\infty} f_i(x) = f(x)$ *for* $x \in [a,b]$. *Show that*

$$V_a^b(f) \leq \liminf_{i\to\infty} V_a^b(f_i).$$

Solution
Let $P_n = \{a = x_0 < x_1 < ... < x_n = b\}$ be a partition of $[a,b]$. Then

$$\begin{aligned}V_a^b(f,P_n) &= \sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|,\\ V_a^b(f_i,P_n) &= \sum_{k=1}^{n}|f_i(x_k)-f_i(x_{k-1})| \text{ for each } i\in\mathbb{N}.\end{aligned}$$

Consider the counting measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ where ν is the counting measure. Let $D = \{1,2,...,n\}$. Then $D \in \mathcal{P}(\mathbb{N})$. Define

$$\begin{aligned}g_i(k) &= |f_i(x_k)-f_i(x_{k-1})| \geq 0,\\ g(k) &= |f(x_k)-f(x_{k-1})| \text{ for } k\in D.\end{aligned}$$

Since $\lim_{i\to\infty} f_i(x) = f(x)$ for $x \in [a,b]$, we have

$$\lim_{i\to\infty} g_i(k) = g(k) \quad \text{for every} \quad k \in D.$$

By Fatou's lemma,

$$\int_D g(k)d\nu = \int_D \lim_{i\to\infty} g_i(k)d\nu \leq \liminf_{i\to\infty} \int_D g_i(k)d\nu. \quad (*)$$

Since $D = \bigsqcup_{k=1}^{n}\{k\}$ (union of disjoint sets), we have

$$\begin{aligned}\int_D g(k)d\nu &= \sum_{k=1}^{n} \int_{\{k\}} g(k)d\nu \\ &= \sum_{k=1}^{n} g(k) \\ &= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\ &= V_a^b(f, P_n).\end{aligned}$$

Similarly, we get

$$\int_D g_i(k)d\nu = V_a^b(f_i, P_n) \quad \text{for each} \quad i \in \mathbb{N}.$$

With these, we can rewrite (*) as follows:

$$V_a^b(f, P_n) \leq \liminf_{i\to\infty} V_a^b(f_i, P_n).$$

By taking all partitions P_n, we obtain

$$V_a^b(f) \leq \liminf_{i\to\infty} V_a^b(f_i). \quad \blacksquare$$

Problem 97
Let f be a real-valued absolutely continuous function on $[a,b]$. If f is never zero, show that $\frac{1}{f}$ is also absolutely continuous on $[a,b]$.

Solution
The function f is continuous on $[a,b]$, which is compact, so f has a minimum on it. Since f is non-zero, there is some $m \in (0,\infty)$ such that

$$\min_{x\in[a,b]} |f(x)| = m.$$

Given any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite family of non-overlapping closed intervals $\{[a_i, b_i] : i = 1, ..., n\}$ in $[a, b]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$ we have $\sum_{i=1}^{n} |f(a_i) - f(b_i)| < \varepsilon$. Now,

$$\begin{aligned}\sum_{i=1}^{n}\left|\frac{1}{f(a_i)} - \frac{1}{f(b_i)}\right| &= \sum_{i=1}^{n}\frac{|f(a_i) - f(b_i)|}{|f(a_i)f(b_i)|} \\ &\leq \frac{1}{m^2}\sum_{i=1}^{n}|f(a_i) - f(b_i)| \\ &\leq \frac{\varepsilon}{m^2}. \quad \blacksquare\end{aligned}$$

Problem 98
Let f be a real-valued function on $[a, b]$ satisfying the Lipschitz condition on $[a, b]$. Show that f is absolutely continuous on $[a, b]$.

Solution
The Lipschitz condition on $[a, b]$:

$$\exists K > 0 : \forall x, y \in [a, b],\ |f(x) - f(y)| \ \leq K|x - y|.$$

Given any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{K}$. Let $\{[c_i, d_i] : i = 1, ..., n\}$ be a family of non-overlapping subintervals of $[a, b]$ with $\sum_{i=1}^{n}(d_i - c_i) < \delta$, then, by the Lipschitz condition, we have

$$\begin{aligned}\sum_{i=1}^{n}|f(c_k) - f(d_k)| &\leq \sum_{i=1}^{n}K(d_k - c_k) \\ &\leq K\sum_{i=1}^{n}(d_k - c_k) \\ &< K\frac{\varepsilon}{K} = \varepsilon.\end{aligned}$$

Thus f is absolutely continuous on $[a, b]$. $\blacksquare$

Problem 99
Show that if f is continuous on $[a, b]$ and f' exists on (a, b) and satisfies $|f'(x)| \leq M$ for $x \in (a, b)$ with some $M > 0$, then f satisfies the Lipschitz condition and thus absolutely continuous on $[a, b]$.

(Hint: Just use the Intermediate Value Theorem.)

Problem 100

Let f be a continuous function on $[a, b]$. Suppose f' exists on (a, b) and satisfies $|f'(x)| \leq M$ for $x \in (a, b)$ with some $M > 0$. Show that for every $E \subset [a, b]$ we have

$$\mu_L^*\big(f(E)\big) \leq M\mu_L^*(E).$$

Solution

Recall:

$$\mu_L^*(E) = \inf\left\{\sum_{n=1}^{\infty}\ell(I_n) : \ I_n \ \text{ are open intervals and } \ \bigcup_{n=1}^{\infty} I_n \supset E\right\}.$$

Let $E \subset [a, b]$. Let $\{I_n = (a'_n, b'_n)\}$ be a covering of E, where each $(a'_n, b'_n) \subset [a, b]$. Then

$$E \subset \bigcup_n (a'_n, b'_n) \ \Rightarrow \ f(E) \subset \bigcup_n f\big((a'_n, b'_n)\big).$$

Since f is continuous, $f\big((a'_n, b'_n)\big)$ must be an interval. So

$$f\big((a'_n, b'_n)\big) = \big(f(a_n), f(b_n)\big) \ \text{ for } \ a_n, b_n \in (a'_n, b'_n).$$

Hence,

$$f(E) \subset \bigcup_n \big(f(a_n), f(b_n)\big).$$

Therefore $\{\big(f(a_n), f(b_n)\big)\}$ is a covering of $f(E)$. By the Mean Value Theorem,

$$\begin{aligned}
\ell\Big(\big(f(a_n), f(b_n)\big)\Big) &= |f(b_n) - f(a_n)| \\
&= |f'(x)||b_n - a_n|, \quad x \in (a_n, b_n) \\
&\leq M|b_n - a_n|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_n \ell\Big(\big(f(a_n), f(b_n)\big)\Big) &\leq M\sum_n |b_n - a_n| \leq M\sum_n |b'_n - a'_n| \\
&\leq M\sum_n \ell\big((a'_n, b'_n)\big).
\end{aligned}$$

Thus,

$$\inf \sum_n \ell\Big(\big(f(a_n), f(b_n)\big)\Big) \leq M \inf \sum_n \ell\big((a'_n, b'_n)\big).$$

The infimum is taken over coverings of $f(E)$ and E respectively. By definition (at the very first of the proof) we have

$$\mu_L^*\big(f(E)\big) \leq M \mu_L^*(E). \quad \blacksquare$$

Problem 101
Let f be a real-valued function on $[a, b]$ such that f is absolutely continuous on $[a + \eta, b]$ for every $\eta \in (0, b - a)$. Show that if f is continuous and of bounded variation on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Solution
Using the Banach-Zaracki theorem, to show that f is absolutely continuous on $[a, b]$, we need to show that f has property (N) on $[a, b]$. Suppose $E \subset [a, b]$ such that $\mu_L(E) = 0$. Given any $\varepsilon > 0$, since f is continuous at a^+, there exists $\delta \in (0, b - a)$ such that

$$a \leq x \leq a + \delta \ \Rightarrow \ |f(x) - f(a)| < \frac{\varepsilon}{2}. \qquad (*)$$

Let $E_1 = E \cap [a, a + \delta]$ and $E_2 = E \setminus E_1$. Then $E = E_1 \cup E_2$ and so $f(E) = f(E_1) \cup f(E_2)$. But $E_2 \subset [a + \delta, b)$ and f is absolutely continuous on $[a + \delta, b]$, so f has property (N) on this interval. Since $E_2 \subset E$, we have $\mu_L(E_2) = 0$. Therefore,

$$\mu_L(f(E_2)) = 0 = \mu_L^*(f(E_2)).$$

On the other hand,

$$\begin{aligned} x \in E_1 \ &\Rightarrow \ x \in [a, a + \delta) \\ &\Rightarrow \ f(a) - \frac{\varepsilon}{2} \leq f(x) \leq f(a) + \frac{\varepsilon}{2} \ \text{ by } \ (*) \\ &\Rightarrow \ f(E_1) \subset [\ f(a) - \frac{\varepsilon}{2}, \ f(a) + \frac{\varepsilon}{2}] \\ &\Rightarrow \ \mu_L^*(f(E_1)) \leq \varepsilon. \end{aligned}$$

Thus,

$$\mu_L^*(f(E)) \leq \mu_L^*(f(E_1)) + \mu_L^*(f(E_2)) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\mu_L^*(f(E)) = 0$ and so $\mu_L(f(E)) = 0$. $\blacksquare$

Problem 102
Let f be a real-valued integrable function on $[a,b]$. Let

$$F(x) = \int_{[a,x]} f d\mu_L, \quad x \in [a,b].$$

Show that F is continuous and of bounded variation on $[a,b]$.

Solution
The continuity follows from Theorem 18 (absolute continuity implies continuity).
To show that F is of BV on $[a,b]$, let $a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a,b]$. Then

$$\begin{aligned} \sum_{i=1}^{n} |F(x_i - x_{i-1}| &= \sum_{i=1}^{n} \left| \int_{[x_{i-1},x_i]} f d\mu_L \right| \\ &\leq \sum_{i=1}^{n} \int_{[x_{i-1},x_i]} |f| d\mu_L \\ &= \int_{[a,b]} |f| d\mu_L. \end{aligned}$$

Thus, since $|f|$ is integrable,

$$V_a^b(F) \leq \int_{[a,b]} |f| d\mu_L < \infty. \quad \blacksquare$$

Chapter 10

L^p Spaces

1. Norms
For $0 < p < \infty$:

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}.$$

For $p = \infty$:

$$\|f\|_\infty = \inf\big\{ M \in [0, \infty) : \ \mu\{x \in X : \ |f(x)| > M\} = 0 \big\}.$$

Theorem 22 *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then the linear space $L^p(X)$ is a Banach space with respect to the norm $\|.\|_p$ for $1 \leq p < \infty$ or the norm $\|.\|_\infty$ for $p = \infty$.*

2. Inequalities for $1 \leq p < \infty$

1. Hölder's inequality: If p and q satisfy the condition $\frac{1}{p} + \frac{1}{q} = 1$, then for $f \in L^p(X),\ g \in L^q(X)$, we have

$$\int_X |fg| d\mu = \left(\int_X |f|^p d\mu \right)^{1/p} \left(\int_X |g|^q d\mu \right)^{1/q},$$

or

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

In particular,

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2 \ \ \text{(Schwarz's inequality)}.$$

2. Minkowski's inequality: For $f, g \in L^p(X)$, we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

3. Convergence

Theorem 23 *Let (f_n) be a sequence in $L^p(X)$ and f an element in $L^p(X)$ with $1 \leq p < \infty$. If $f_n \to f$ in $L^p(X)$, i.e., $\|f_n - f\|_p \to 0$, then*
(1) $\|f_n\|_p \to \|f\|_p$,
(2) $f_n \xrightarrow{\mu} f$ on X,
(3) There exists a subsequence (f_{n_k}) such that $f_{n_k} \to f$ a.e. on X.

Theorem 24 *Let* (f_n) *be a sequence in* $L^p(X)$ *and* f *an element in* $L^p(X)$ *with* $1 \leq p < \infty$. *If* $f_n \to f$ *a.e. on* X *and* $\|f_n\|_p \to \|f\|_p$, *then* $\|f_n - f\|_p \to 0$.

Theorem 25 *Let* (f_n) *be a sequence in* $L^p(X)$ *and* f *an element in* $L^p(X)$ *with* $1 \leq p < \infty$. *If* $f_n \xrightarrow{\mu} f$ *on* X *and* $\|f_n\|_p \to \|f\|_p$, *then* $\|f_n - f\|_p \to 0$.

Theorem 26 *Let* (f_n) *be a sequence in* $L^p(X)$ *and* f *an element in* $L^p(X)$ *with* $1 \leq p < \infty$. *If* $\|f_n - f\|_\infty \to 0$, *then*
(1) $\|f_n\|_\infty \to \|f\|_\infty$,
(2) $f_n \to f$ *uniformly on* $X \setminus E$ *where* E *is a null set.*
(3) $f_n \xrightarrow{\mu} f$ *on* X.

Problem 103
Let f *be a Lebesgue measurable function on* $[0,1]$. *Suppose* $0 < f(x) < \infty$ *for all* $x \in [0,1]$. *Show that*

$$\left(\int_{[0,1]} f d\mu\right)\left(\int_{[0,1]} \frac{1}{f} d\mu\right) \geq 1.$$

Solution
The functions $\sqrt{f}$ and $\frac{1}{\sqrt{f}}$ are Lebesgue measurable since f is Lebesgue measurable and $0 < f < \infty$. By Schwarz's inequality, we have

$$\begin{aligned} 1 = \int_{[0,1]} 1 d\mu = \int_{[0,1]} \sqrt{f}\ \frac{1}{\sqrt{f}} d\mu &\leq \left(\int_{[0,1]} \left(\sqrt{f}\right)^2 d\mu\right)^{1/2} \left(\int_{[0,1]} \left(\frac{1}{\sqrt{f}}\right)^2 d\mu\right)^{1/2} \\ &\leq \left(\int_{[0,1]} f d\mu\right)^{1/2} \left(\int_{[0,1]} \frac{1}{f} d\mu\right)^{1/2}. \end{aligned}$$

Squaring both sides we get

$$\left(\int_{[0,1]} f d\mu\right)\left(\int_{[0,1]} \frac{1}{f} d\mu\right) \geq 1. \quad \blacksquare$$

Problem 104
Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $f \in L^p(X)$ with $p \in (1, \infty)$ and q its conjugate. Show that

$$\int_X |f| d\mu \leq \mu(X)^{\frac{1}{q}} \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}.$$

Hint:
Write

$$f = f \mathbf{1}_X$$

where $\mathbf{1}_X$ is the characteristic function of X, then apply the Hölder's inequality.

Problem 105
Let $(X, \mathcal{A}, \mu)$ be a finite measure space.
(1) If $1 \leq p < \infty$ show that $L^\infty(X) \subset L^p(X)$.
(2) If $1 \leq p_1 < p_2 < \infty$ show that $L^{p_2}(X) \subset L^{p_1}(X)$.

Solution

(1) Take any $f \in L^\infty(X)$. Then $\|f\|_\infty < \infty$. By definition, we have $|f| \leq \|f\|_\infty$ a.e. on X. So we have

$$\int_X |f|^p d\mu \leq \int_X \|f\|_\infty^p d\mu = \mu(X) \|f\|_\infty^p.$$

By assumption, $\mu(X) < \infty$. Thus $\int_X |f|^p d\mu < \infty$. That is $f \in L^p(X)$.

(2) Consider the case $1 \leq p_1 < p_2 < \infty$. Take any $f \in L^{p_2}(X)$. Let $\alpha := \frac{p_2}{p_1}$. Then $1 < \alpha < \infty$. Let $\beta \in (1, \infty)$ be the conjugate of α, that is, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. By the Hölder's inequality, we have

$$\begin{aligned}
\int_X |f|^{p_1} d\mu &= \int_X \left(|f|^{p_2}\right)^{1/\alpha} \mathbf{1}_X d\mu \\
&\leq \left(\int_X |f|^{p_2} d\mu \right)^{1/\alpha} \left(\int_X |\mathbf{1}_X|^\beta d\mu \right)^{1/\beta} \\
&= \|f\|_{p_2}^{p_2/\alpha} \mu(X) < \infty,
\end{aligned}$$

since $\|f\|_{p_2} < \infty$ and $\mu(X) < \infty$. Thus $f \in L^{p_1}(X)$. ■

Problem 106 (Extension of Hölder's inequality)
Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space. Let $f_1, ..., f_n$ be extended complex-valued measurable functions on X such that $|f_1|, ..., |f_n| < \infty$ a.e. on X. Let $p_1, ..., p_n$ be real numbers such that

$$p_1, ..., p_n \in (1, \infty) \quad \textit{and} \quad \frac{1}{p_1} + ... + \frac{1}{p_n} = 1.$$

Prove that

$$\|f_1 ... f_n\|_1 \leq \|f_1\|_{p_1} ... \|f_n\|_{p_n}. \qquad (*)$$

Hint:
Proof by induction. For $n = 2$ we have already the Hölder's inequality.
Assume that $(*)$ holds for $n \geq 2$. Let

$$q = \left(\frac{1}{p_1} + ... + \frac{1}{p_n}\right)^{-1}.$$

Then

$$q, p_{n+1} \in (0, \infty) \text{ and } \frac{1}{q} + \frac{1}{p_{n+1}} = 1.$$

Keep going this way.

Problem 107
Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space. Let $f_1, ..., f_n$ be extended complex-valued measurable functions on X such that $|f_1|, ..., |f_n| < \infty$ a.e. on X. Let $p_1, ..., p_n$ and r be real numbers such that

$$p_1, ..., p_n, r \in (1, \infty) \quad \textit{and} \quad \frac{1}{p_1} + ... + \frac{1}{p_n} = \frac{1}{r}. \qquad (i)$$

Prove that

$$\|f_1 ... f_n\|_r \leq \|f_1\|_{p_1} ... \|f_n\|_{p_n}.$$

Solution

We can write (i) as follows:

$$\frac{1}{p_1/r} + ... + \frac{1}{p_n/r} = 1.$$

From the extension of Hölder's inequality (Problem 105) we have

$$\big\||f_1...f_n|^r\big\|_1 \leq \big\||f_1|^r\big\|_{p_1/r}...\big\||f_n|^r\big\|_{p_n/r}. \quad (ii)$$

Now we have

$$\big\||f_1...f_n|^r\big\|_1 = \int_X |f_1...f_n|^r d\mu = \|f_1...f_n\|_r^r,$$

and for $i = 1, ..., n$ we have

$$\big\||f_i|^r\big\|_{p_i/r} = \left(\int_X |f_i|^{r\frac{p_i}{r}} d\mu\right)^{r/p_i} = \left(\int_X |f_i|^{p_i} d\mu\right)^{r/p_i} = \|f_i\|_{p_i}^r.$$

By substituting these expressions into (ii), we have

$$\|f_1...f_n\|_r^r \leq \|f_1\|_{p_1}^r...\|f_n\|_{p_n}^r.$$

Taking the r-th roots both sides of the above inequality we obtain (i). ■

Problem 108
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\theta \in (0,1)$ and let $p, q, r \geq 1$ with $p, q \geq r$ be related by

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

Show that for every extended complex-valued measurable function on X we have

$$\|f\|_r \leq \|f\|_p^{\theta}\|f\|_q^{1-\theta}.$$

Solution
Recall: (Extension of Holder's inequality)

$$\frac{1}{r} = \frac{1}{p_1} + ... + \frac{1}{p_n} \;\Rightarrow\; \|f_1...f_n\|_r \leq \|f\|_{p_1}...\|f_n\|_{p_n}.$$

For $n = 2$ we have

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \;\Rightarrow\; \|fg\|_r \leq \|f\|_p\|g\|_q.$$

Now, we have

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q} = \frac{1}{p/\theta} + \frac{1}{q/(1-\theta)}.$$

Applying the above formula we get

$$\|f\|_r = \left\||f|^\theta . |f|^{1-\theta}\right\| \leq \left\||f|^\theta\right\|_{p/\theta} . \left\||f|^{1-\theta}\right\|_{q/(1-\theta)} . \quad (*)$$

Some more calculations:

$$\begin{aligned} \left\||f|^\theta\right\|_{p/\theta} &= \left(\int_X \left(|f|^\theta\right)^{p/\theta}\right)^{\theta/p} \\ &= \left(\int_X |f|^p\right)^{\theta/p} \\ &= \|f\|_p^\theta . \end{aligned}$$

And

$$\begin{aligned} \left\||f|^{1-\theta}\right\|_{q/(1-\theta)} &= \left(\int_X \left(|f|^{1-\theta}\right)^{q/1-\theta}\right)^{1-\theta/q} \\ &= \left(\int_X |f|^q\right)^{1-\theta/q} \\ &= \|f\|_q^{1-\theta} . \end{aligned}$$

Plugging into (*) we obtain

$$\|f\|_r \leq \|f\|_p^\theta . \|f\|_q^{1-\theta} . \quad \blacksquare$$

Problem 109

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p, q \in [1, \infty]$ be conjugate. Let $(f_n)_{n\in\mathbb{N}} \subset L^p(X)$ and $f \in L^p(X)$ and similarly $(g_n)_{n\in\mathbb{N}} \subset L^q(X)$ and $g \in L^q(X)$. Show that

$$\left[\lim_{n\to\infty} \|f_n - f\|_p = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|g_n - g\|_q = 0\right] \Rightarrow \lim_{n\to\infty} \|f_n g_n - fg\|_1 = 0.$$

Solution

We use Hölder's inequality:

$$\begin{aligned} \|f_n g_n - fg\|_1 &= \int_X |f_n g_n - fg| d\mu \\ &\leq \int_X (|f_n g_n - f_n g| + |f_n g - fg|) d\mu \\ &\leq \int_X |f_n||g_n - g| d\mu + \int_X |g||f_n - f| d\mu \\ &\leq \|f_n\|_p . \|g_n - g\|_q + \|g\|_q . \|f_n - f\|_p . \quad (*) \end{aligned}$$

By Minkowski's inequality, we have

$$\|f_n\|_p \leq \|f\|_p + \|f_n - f\|_p.$$

Since $\|f\|_p$ and $\|f_n - f\|_p$ are bounded (why?), $\|f_n\|_p$ is bounded for every $n \in \mathbb{N}$. From assumptions we deduce that $\lim_{n\to\infty} \|f_n\|_p.\|g_n - g\|_q = 0$.
Since $\|g\|_q$ is bounded, from assumptions we get $\lim_{n\to\infty} \|g\|_q.\|f_n - f\|_p = 0$. Therefore, from (*) we obtain

$$\lim_{n\to\infty} \|f_n g_n - fg\|_1 = 0. \quad \blacksquare$$

Problem 110
Let $(X, \mathcal{A}, \mu)$ be a measure space and let $p \in [1, \infty)$. Let $(f_n)_{n\in\mathbb{N}} \subset L^p(X)$ and $f \in L^p(X)$ be such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued measurable functions on X such that $|g_n| \leq M$ for every $n \in \mathbb{N}$ and let g be a complex-valued measurable function on X such that $\lim_{n\to\infty} g_n = g$ a.e. on X. Show that

$$\lim_{n\to\infty} \|f_n g_n - fg\|_p = 0.$$

Solution
We first note that $|g| \leq M$ a.e. on X. Indeed, we have for all $n \in \mathbb{N}$,

$$|g| \leq |g_n - g| + |g_n|.$$

Since $|g_n| \leq M$ for every $n \in \mathbb{N}$ and $|g_n - g| \to 0$ a.e. on X by assumption. Hence $|g| \leq M$ a.e. on X.
Now, by Minkowski's inequality, we have

$$\begin{aligned} \|f_n g_n - fg\|_p &\leq \|f_n g_n - f g_n\|_p + \|f g_n - fg\|_p \\ &\leq \|g_n(f_n - f)\|_p + \|f(g_n - g)\|_p \quad (*) \end{aligned}$$

Some more calculations:

$$\begin{aligned} \|g_n(f_n - f)\|_p^p &= \int_X |g_n(f_n - f)|^p d\mu \\ &\leq \int_X |g_n|^p.|f_n - f|^p d\mu \\ &\leq M^p \|f_n - f\|_p^p. \end{aligned}$$

Since $\|f_n - f\|_p \to 0$ by assumption, we have that $\|g_n(f_n - f)\|_p \to 0$.
Let $h_n = fg_n - fg$ for every $n \in \mathbb{N}$. Then

$$\begin{aligned}|h_n| &\leq |f|.|g_n - g| \leq |f|(|g_n| + |g|) \leq 2M|f| \\ |h_n|^p &\leq 2^p M^p |f|^p < \infty.\end{aligned}$$

Now, $|h_n|^p$ is bounded and $|h_n|^p \leq |f|^p.|g_n - g|^p \Rightarrow |h_n|^p \to 0$ (since $g_n \to g$ a.e.). By the Dominated Convergence Theorem, we have

$$\begin{aligned}0 = \int_X \lim_{n\to\infty} |h_n|^p d\mu &= \lim_{n\to\infty} \int_X |h_n|^p d\mu \\ &= \lim_{n\to\infty} \int_X |fg_n - fg|^p d\mu \\ &= \lim_{n\to\infty} \|f(g_n - g)\|_p^p.\end{aligned}$$

From these results, (*) gives that

$$\lim_{n\to\infty} \|f_n g_n - fg\|_p = 0. \quad \blacksquare$$

Problem 111
Let f be an extended real-valued Lebesgue measurable function on $[0,1]$ such that $\int_{[0,1]} |f|^p d\mu < \infty$ for some $p \in [1, \infty)$. Let $q \in (1, \infty]$ be the conjugate of p. Let $a \in (0,1]$. Show that

$$\lim_{a\to 0} \frac{1}{a^{1/q}} \int_{[0,a]} |f| d\mu = 0.$$

Solution

• $\underline{p = 1}$
Since $q = \infty$, we have to show

$$\lim_{a\to 0} \int_0^a |f(s)| ds = 0 \quad \text{(Lebesgue integral = Riemann integral)}.$$

This is true since f is integrable so $\int_0^a |f(s)| ds$ is continuous with respect to a.

• $\underline{1 < p < \infty}$
Then $1 < q < \infty$. We have

$$\begin{aligned}\int_0^a |f(s)| ds &= \int_0^a |f(s)|.1 \, ds \\ &\leq \mu([0,a])^{1/q} \left(\int_0^a |f(s)| ds \right)^{1/p} \quad \text{(Problem 104)} \\ &= a^{1/q} \left(\int_0^a |f(s)| ds \right)^{1/p}.\end{aligned}$$

Hence,

$$\frac{1}{a^{1/q}}\int_0^a |f(s)|ds \leq \left(\int_0^a |f(s)|ds\right)^{1/p} \qquad (*)$$

Since $|f|$ is integrable, we have[1] (Problem 66)

$$\forall\varepsilon>0,\ \exists\delta>0:\ \mu([0,a])<\delta\ \Rightarrow\ \int_{[0,a]}|f|d\mu<\varepsilon^p.$$

Equivalently,

$$\forall\varepsilon>0,\ \exists\delta>0:\ 0<a<\delta\Rightarrow\ \left(\int_0^a |f(s)|ds\right)^{1/p}<\varepsilon. \quad (**)$$

From (*) and (**) we obtain

$$\forall\varepsilon>0,\ \exists\delta>0:\ 0<a<\delta\Rightarrow\ \frac{1}{a^{1/q}}\int_0^a |f(s)|ds<\varepsilon.$$

That is,

$$\lim_{a\to 0}\frac{1}{a^{1/q}}\int_0^a |f(s)|ds=0. \quad \blacksquare$$

Problem 112
Let $(X,\mathcal{A},\mu)$ be a finite measure space. Let $f_n, f\in L^2(X)$ for all $n\in\mathbb{N}$ such that $\lim_{n\to\infty} f_n=f$ a.e. on X and $\|f_n\|_2\leq M$ for all $n\in\mathbb{N}$.
Show that $\lim_{n\to\infty}\|f_n-f\|_1=0$.

Solution
We first claim: $\|f\|_2\leq M$. Indeed, y Fatous' lemma, we have

$$\|f\|_2^2=\int_X |f|^2d\mu\leq\liminf_{n\to\infty}\int_X |f_n|^2d\mu\leq M^2.$$

Since $\mu(X)<\infty$, we can use Egoroff's theorem:

$$\forall\varepsilon>0,\ \exists A\in\mathcal{A}\ \text{ with }\ \mu(A)<\varepsilon^2\ \text{ and }\ f_n\to f\ \text{ uniformly on }\ X\setminus A.$$

Now we can write

$$\|f_n-f\|_1=\int_X|f_n-f|d\mu=\int_A|f_n-f|d\mu+\int_{X\setminus A}|f_n-f|d\mu.$$

[1]This is called *the uniform continuity of the integral with respect to the measure μ.*

On $X \setminus A$, $f_n \to f$ uniformly, so for large n, we have $\int_{X\setminus A} |f_n - f| d\mu < \varepsilon$. On A we have

$$\begin{aligned}\int_A |f_n - f| d\mu = \int_X |f_n - f|\chi_A \, d\mu &\leq \mu(A)^{1/2}.\|f_n - f\|_2 \\ &\leq \mu(A)^{1/2}(\|f_n\|_2 + \|f\|_2) \\ &\leq 2M\varepsilon \quad (\text{since } \ \mu(A) < \varepsilon^2).\end{aligned}$$

Thus, for any $\varepsilon > 0$, for large n, we have

$$\|f_n - f\|_1 \leq (2M + 1)\varepsilon.$$

This tells us that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. ■

Problem 113
Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $p, q \in (1, \infty)$ be conjugates. Let $f_n, f \in L^p(X)$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} f_n = f$ a.e. on X and $\|f_n\|_p \leq M$ for all $n \in \mathbb{N}$. Show that
(a) $\|f\|_p \leq M$.
(b) $\lim_{n\to\infty} \|f_n - f\|_p = 0$.
(c) $\lim_{n\to\infty} \int_X f_n g d\mu = \int_X f g d\mu$ *for every* $g \in L^q(X)$.
(d) $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$ *for every* $E \in \mathcal{A}$.

Hint:
(a) and (b): See Problem 112.
(c) Show $\|f_n g - fg\|_1 \leq \|f_n - f\|_p \|g\|_q$. Then use (b).
(d) Write

$$\int_E f_n g = \int_X f_n g \mathbf{1}_E = \int_X f_n (g\mathbf{1}_E).$$

Then use (c).

Problem 114
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f be a real-valued measurable function on X such that $f \in L^1(X) \cap L^\infty(X)$. Show that $f \in L^p(X)$ for every $p \in [1, \infty]$.

Hint:
If $p = 1$ or $p = \infty$, there is nothing to prove.
Suppose $p \in (1, \infty)$. Let $f \in L^1(X) \cap L^\infty(X)$. Write

$$|f|^p = |f|^1 |f|^{p-1} \leq |f|.\|f\|_\infty^{p-1}.$$

Integrate over X, then use the fact that $\|f\|_1$ and $\|f\|_\infty$ are finite.

Problem 115
Let $(X, \mathcal{A}, \mu)$ be a measure space and let $0 < p_1 < p < p_2 \leq \infty$. Show that

$$L^p(X) \subset L^{p_1}(X) + L^{p_2}(X),$$

that is, if $f \in L^p(X)$ then $f = g+h$ for some $g \in L^{p_1}(X)$ and some $h \in L^{p_2}(X)$.

Solution
For any $f \in L^p(X)$, let $D = \{X : \ |f| \geq 1\}$. Let $g = f\mathbf{1}_D$ and $h = f\mathbf{1}_{D^c}$. Then

$$g + h = f\mathbf{1}_D + f\mathbf{1}_{D^c} = f(\underbrace{\mathbf{1}_D + \mathbf{1}_{D^c}}_{=\mathbf{1}_{D\cup D^c}}) = f \quad \text{(See Problem 37).}$$

We want to show $g \in L^{p_1}(X)$ and $h \in L^{p_2}(X)$.
On D we have : $1 \leq |f|^{p_1} \leq |f|^p \leq |f|^{p_2}$. It follows that

$$\int_X |g|^{p_1} d\mu = \int_D |f|^{p_1} d\mu \leq \int_X |f|^p d\mu < \infty \ \text{ since } \ f \in L^p(X).$$

Hence, $g \in L^{p_1}(X)$.
On D^c we have : $|f|^{p_1} \geq |f|^p \geq |f|^{p_2}$. It follows that

$$\int_X |h|^{p_2} d\mu = \int_{D^c} |f|^{p_2} d\mu \leq \int_X |f|^p d\mu < \infty.$$

Hence, $h \in L^{p_2}(X)$. This completes the proof. ■

Problem 116
Given a measure space $(X, \mathfrak{A}, \mu)$. For $0 < p < r < q \leq \infty$, show that

$$L^p(X) \cap L^q(X) \subset L^r(X).$$

Hint:
Let $D = \{X : \ |f| \geq 1\}$. On D we have $|f|^r \leq |f|^q$, and on $X \setminus D$ we have $|f|^r \leq |f|^p$.

Problem 117
Suppose $f \in L^4([0,1])$, $\|f\|_4 = C \geq 1$ *and* $\|f\|_2 = 1$. *Show that*

$$\frac{1}{C} \leq \|f\|_{4/3} \leq 1.$$

Solution
First we note that 4 and 4/3 are conjugate. By assumption and by Hölder's inequality we have

$$\begin{aligned} 1 = \|f\|_2^2 = \int_{[0,1]} |f|^2 d\mu &= \int_{[0,1]} |f|.|f| d\mu \\ &\leq \|f\|_4.\|f\|_{4/3} \\ &\leq C.\|f\|_{4/3}. \end{aligned}$$

This implies that $\|f\|_{4/3} \geq \frac{1}{C}$. $(*)$.
By Schwrarz's inequality we have

$$\begin{aligned} \|f\|_{4/3}^{4/3} = \int_{[0,1]} |f|^{4/3} d\mu &= \int_{[0,1]} |f|.|f|^{1/3} d\mu \\ &\leq \|f\|_2.\|f\|_2^{1/3} = 1 \text{ since } \|f\|_2 = 1. \end{aligned}$$

Hence, $\|f\|_{4/3} \leq 1$. $(**)$
From (*) and (**) we obtain

$$\frac{1}{C} \leq \|f\|_{4/3} \leq 1. \quad \blacksquare$$

Problem 118
Let $(X, \mathcal{A}, \mu)$ *be a measure space with* $\mu(X) \in (0, \infty)$. *Let* $f \in L^\infty(X)$ *and let* $\alpha_n = \int_X |f|^n d\mu$ *for* $n \in \mathbb{N}$. *Show that*

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

Solution
We first note that if $\|f\|_\infty = 0$, the problem does not make sense. Indeed,

$$\begin{aligned} \|f\|_\infty = 0 &\Rightarrow f \equiv 0 \ \ a.e. \text{ on } X \\ &\Rightarrow \alpha_n = 0, \ \forall n \in \mathbb{N}. \end{aligned}$$

Suppose that $0 < \|f\|_\infty < \infty$. Then $\alpha_n > 0,\ \forall n \in \mathbb{N}$. We have

$$\begin{aligned}\alpha_{n+1} = \int_X |f|^{n+1} d\mu &= \int_X |f|^n |f| d\mu \\ &\leq \|f\|_\infty . \int_X |f|^n d\mu = \|f\|_\infty \alpha_n .\end{aligned}$$

This implies that

$$\frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty,\ \forall n \in \mathbb{N}.$$

$$\Rightarrow \limsup_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty. \quad (*)$$

Notice that $\frac{n+1}{n}$ and $n+1$ are conjugate. Using again Hölder's inequality, we get

$$\begin{aligned}\alpha_n = \int_X |f|^n .1\ d\mu &\leq \left(\int_X (|f|^n)^{\frac{n+1}{n}} d\mu\right)^{\frac{n}{n+1}} \left(\int_X 1^{n+1}\right)^{\frac{1}{n+1}} \\ &= \left(\int_X (|f|^{n+1}) d\mu\right)^{\frac{n}{n+1}} .\mu(X)^{\frac{1}{n+1}} \\ &= \alpha_{n+1}^{\frac{n}{n+1}} .\mu(X)^{\frac{1}{n+1}}.\end{aligned}$$

With a simple calculation we get

$$\frac{\alpha_{n+1}}{\alpha_n} \geq \alpha_{n+1}^{\frac{1}{n+1}} .\mu(X)^{-\frac{1}{n+1}},\ \forall n \in \mathbb{N}.$$

Given any $\varepsilon > 0$, let $E = \{X :\ |f| > \|f\|_\infty - \varepsilon\}$, then, by definition of $\|f\|_\infty$, we have $\mu(E) > 0$. Now,

$$\begin{aligned}\alpha_{n+1}^{\frac{1}{n+1}} &= \left(\int_X (|f|^{n+1}) d\mu\right)^{\frac{1}{n+1}} \\ &\geq \left(\int_E (|f|^{n+1}) d\mu\right)^{\frac{1}{n+1}} \\ &> \mu(E)^{\frac{1}{n+1}} .(\|f\|_\infty - \varepsilon).\end{aligned}$$

It follows that

$$\frac{\alpha_{n+1}}{\alpha_n} \geq (\|f\|_\infty - \varepsilon). \left[\frac{\mu(E)}{\mu(X)}\right]^{\frac{1}{n+1}}.$$

$$\Rightarrow \liminf_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \geq \|f\|_\infty - \varepsilon,\ \forall \varepsilon > 0$$

$$\Rightarrow \liminf_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \geq \|f\|_\infty. \quad (**)$$

From (*) and (**) we obtain

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty. \quad \blacksquare$$

Problem 119
Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in [1, \infty)$.
Let $f \in L^p(X)$ and $(f_n : n \in \mathbb{N}) \subset L^p(X)$. Suppose $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ we have

$$\int_E |f_n|^p d\mu < \varepsilon \quad \text{for every } E \in \mathcal{A} \text{ such that } \mu(E) < \delta.$$

Solution
By assumption we have $\lim_{n\to\infty} \|f_n - f\|_p^p = 0$. Equivalently,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} :\ n \geq N \Rightarrow \|f_n - f\|_p^p < \frac{\varepsilon}{2^{p+1}}. \quad (1)$$

From triangle inequality we have[2]

$$|f_n| \leq |f_n - f| + |f|,$$
$$|f_n|^p \leq (|f_n - f| + |f|)^p \leq 2^p |f_n - f|^p + 2^p |f|^p.$$

Integrating over $E \in \mathcal{A}$ and using (1), we get for $n \geq N$,

$$\begin{aligned}
\int_E |f_n|^p d\mu &\leq 2^p \int_E |f_n - f|^p d\mu + 2^p \int_E |f|^p d\mu \\
&\leq 2^p \|f_n - f\|_p^p + 2^p \int_E |f|^p d\mu \\
&\leq 2^p . \frac{\varepsilon}{2^{p+1}} + 2^p \int_E |f|^p d\mu \\
&= \frac{\varepsilon}{2} + 2^p \int_E |f|^p d\mu. \qquad (2)
\end{aligned}$$

[2] In fact, for $a, b \geq 0$ and $1 \leq p < \infty$ we have

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

Since $|f|^p$ is integrable, by *the uniform absolute continuity of integral (Problem 66)* we have

$$\exists \delta_0 > 0 : \ \mu(E) < \delta_0 \Rightarrow \int_E |f|^p d\mu < \frac{\varepsilon}{2^{p+1}}.$$

So, for $n \geq N$, from (2) we get

$$\exists \delta_0 > 0 : \ \mu(E) < \delta_0 \Rightarrow \int_E |f_n|^p d\mu \leq \frac{\varepsilon}{2} + 2^p . \frac{\varepsilon}{2^{p+1}} = \varepsilon. \quad (3)$$

Similarly, all $|f_1|^p, ..., |f_{N-1}|^p$ are integrable, so we have

$$\exists \delta_j > 0 : \ \mu(E) < \delta_j \Rightarrow \int_E |f_j|^p d\mu < \varepsilon, \quad j = 1, ..., N-1. \quad (4)$$

Let $\delta = \min\{\delta_0, \delta_1, .., \delta_{N-1}\}$. From (3) and (4) we get for every $n \in \mathbb{N}$,

$$\exists \delta > 0 : \ \mu(E) < \delta \Rightarrow \int_E |f_n|^p d\mu < \varepsilon. \quad \blacksquare$$

Problem 120
Let f be a bounded real-valued integrable function on $[0,1]$. Suppose $\int_{[0,1]} x^n f d\mu = 0$ for $n = 0, 1, 2, ...$ Show that $f = 0$ a.e. on $[0,1]$.

Solution
Fix an arbitrary function $\varphi \in C[0,1]$. By the Stone-Weierstrass theorem, there is a sequence (p_n) of polynomials such that $\|\varphi - p_n\|_\infty \to 0$ as $n \to \infty$. Then

$$\begin{aligned} \left| \int_{[0,1]} f\varphi d\mu \right| &= \left| \int_{[0,1]} f(\varphi - p_n + p_n) d\mu \right| \\ &\leq \int_{[0,1]} |f| \ |\varphi - p_n| d\mu + \left| \int_{[0,1]} f p_n d\mu \right| \\ &\leq \|f\|_1 \|\varphi - p_n\|_\infty + \underbrace{\left| \int_{[0,1]} f p_n d\mu \right|}_{=0 \text{ by hypothesis}} \\ &= \|f\|_1 \|\varphi - p_n\|_\infty. \end{aligned}$$

Since $\|f\|_1 < \infty$ and $\|\varphi - p_n\|_\infty \to 0$, we have

$$\int_{[0,1]} f\varphi d\mu = 0, \ \forall \varphi \in C[0,1]. \quad (*)$$

Now, since $C[0,1]$ is dense in $L^1[0,1]$, there exists a sequence $(\varphi_n) \subset C[0,1]$ such that $\|\varphi_n - f\|_1 \to 0$ as $n \to \infty$. Then

$$\begin{aligned} 0 \leq \int_{[0,1]} f^2 d\mu &= \left| \int_{[0,1]} f(f - \varphi_n + \varphi_n) d\mu \right| \\ &\leq \int_{[0,1]} |f|\,|f - \varphi_n| d\mu + \underbrace{\left| \int_{[0,1]} f \varphi_n d\mu \right|}_{=0 \text{ by } (*)} \\ &\leq \|f\|_\infty \|f - \varphi_n\|_1. \end{aligned}$$

Since $\|f\|_\infty < \infty$ and $\|f - \varphi_n\|_1 \to 0$, we have

$$\int_{[0,1]} f^2 d\mu = 0.$$

Thus $f = 0$ a.e. on $[0,1]$. ■

Problem 121
Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space with $\mu(X) = \infty$.
(a) Show that there exists a disjoint sequence $(E_n : n \in \mathbb{N})$ in $\mathcal{A}$ such that $\bigcup_{n \in \mathbb{N}} E_n = X$ and $\mu(E_n) \in [1, \infty)$ for every $n \in \mathbb{N}$.
(b) Show that there exists an extended real-valued measurable function f on X such that $f \notin L^1(X)$ and $f \in L^p(X)$ for all $p \in (1, \infty]$.

Solution
(a) Since $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, there exists a sequence $(A_n : n \in \mathbb{N})$ of disjoint sets in $\mathcal{A}$ such that

$$X = \bigcup_{n \in \mathbb{N}} A_n \text{ and } \mu(A_n) < \infty, \forall n \in \mathbb{N}.$$

By the countable additivity and by assumption, we have

$$\mu(X) = \sum_{n \in \mathbb{N}} \mu(A_n) = \infty.$$

It follows that

$$\exists k_1 \in \mathbb{N} : 1 \leq \sum_{n=1}^{k_1} \mu(A_n) = \mu(A_1 \cup ... \cup A_{k_1}) < \infty.$$

Let $E_1 = A_1 \cup ... \cup A_{k_1}$ then we have

$$1 \leq \mu(E_1) < \infty \quad \text{and} \quad \mu(A_{k_1+1} \cup A_{k_1+2} \cup ...) = \mu(X \setminus E_1) = \infty.$$

Then there exists $k_2 \geq k_1 + 1$ such that

$$1 \leq \mu(A_{k_1+1} \cup ... \cup A_{k_2}) < \infty.$$

Let $E_1 = A_{k_1+1} \cup ... \cup A_{k_2}$ then we have

$$1 \leq \mu(E_2) < \infty \quad \text{and} \quad E_1 \cap E_1 = \emptyset.$$

And continuing this process we are building a sequence $(E_n : \; n \in \mathbb{N})$ of disjoint subsets in $\mathcal{A}$ satisfying

$$\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} A_n = X \quad \text{and} \quad \mu(E_n) \in [1, \infty), \; \forall n \in \mathbb{N}.$$

(b) Define a real-valued function f on $X = \bigcup_{n\in\mathbb{N}} A_n$ by

$$f = \sum_{n=1}^{\infty} \frac{\chi_{A_n}}{n\mu(A_n)}.$$

Then

$$f|_{A_1} = \frac{1}{1\mu(A_1)}, ..., f|_{A_n} = \frac{1}{n\mu(A_n)}, ...$$

Hence,

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

That is $f \notin L^1(X)$.
We also have

$$f^p|_{A_1} = \frac{1}{1^p \mu(A_1)^p}, ..., f^p|_{A_n} = \frac{1}{n^p \mu(A_n)^p}, ...(1 < p < \infty)$$

By integrating

$$\begin{aligned} \int_X f^p d\mu &= \sum_{n=1}^{\infty} \int_{A_n} f^p d\mu \\ &= \sum_{n=1}^{\infty} \frac{1}{n^p \mu(A_n)^{p-1}} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty. \;\text{ since }\; \mu(A_n)^{p-1} \geq 1. \end{aligned}$$

Thus, $f \in L^p(X)$. ■

Problem 122
Consider the space $L^p([0,1])$ where $p \in (1,\infty]$.
(a) Prove that $\|f\|_p$ is increasing in p for any bounded measurable function f.
(b) Prove that $\|f\|_p \to \|f\|_\infty$ when $p \to \infty$.

Solution
(a)
• Suppose $1 < p < \infty$. We want to show $\|f\|_p \leq \|f\|_\infty$.
By definition, we have

$$|f| \leq \|f\|_\infty \text{ } a.e. \text{ on } [0,1].$$

Therefore,

$$\begin{aligned} & |f|^p \leq \|f\|_\infty^p \text{ } a.e. \text{ on } [0,1]. \\ & \Rightarrow \int_{[0,1]} |f|^p d\mu \leq \int_{[0,1]} \|f\|_\infty^p d\mu \\ & \Rightarrow \|f\|_p^p \leq \|f\|_\infty^p \, \mu([0,1]) \\ & \Rightarrow \|f\|_p \leq \|f\|_\infty. \end{aligned}$$

• Suppose $1 < p_1 < p_2 < \infty$. We want to show $\|f\|_{p_1} \leq \|f\|_{p_2}$.
Notice that

$$\frac{p_1}{p_2} + \frac{p_2 - p_1}{p_2} = 1 \text{ or } \frac{1}{p_2/p_1} + \frac{1}{p_2/(p_2 - p_1)} = 1.$$

By Hölder's inequality we have

$$\begin{aligned} \|f\|_{p_1}^{p_1} = \int_{[0,1]} |f|^{p_1} d\mu &= \int_{[0,1]} |f|^{p_1}.1.d\mu \\ &\leq \||f|^{p_1}\|_{p_2/p_1}.\|1\|_{p_2/(p_2-p_1)} \\ &= \|f\|_{p_2/p_1}^{p_1}. \quad (*) \end{aligned}$$

Now,

$$\begin{aligned} \|f\|_{p_2/p_1}^{p_1} &= \left(\int_{[0,1]} |f|^{p_1 \cdot \frac{p_2}{p_1}} d\mu \right)^{p_1/p_2} \\ &= \left(\int_{[0,1]} |f|^{p_2} d\mu \right)^{p_1 \cdot \frac{1}{p_2}} = \|f\|_{p_2}^{p_1}. \end{aligned}$$

Finally, (*) implies that $\|f\|_{p_1} \leq \|f\|_{p_2}$.
In both cases we have

$$1 < p_1 < p_2 \Longrightarrow \|f\|_{p_1} \leq \|f\|_{p_2}.$$

That is $\|f\|_p$ is increasing in p.

(b) By part (a) we get $\|f\|_p \le \|f\|_\infty$. Then

$$\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty. \quad (i)$$

Given any $\varepsilon > 0$, let $E = \{X : \ |f| > \|f\|_\infty - \varepsilon\}$. Then $\mu(E) > 0$ and

$$\begin{aligned}
&\|f\|_p^p \ge \int_E |f|^p d\mu > (\|f\|_\infty - \varepsilon)^p \mu(E).\\
&\Rightarrow \|f\|_p \ge (\|f\|_\infty - \varepsilon)\mu(E)^{1/p}\\
&\Rightarrow \liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty - \varepsilon, \ \forall\varepsilon > 0 \ \ (\text{since} \ \ \lim_{p\to\infty} \mu(E)^{1/p} = 1).\\
&\Rightarrow \liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty. \quad (ii)
\end{aligned}$$

From (i) and (ii) we obtain

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty. \quad \blacksquare$$

* * **

APPENDIX

The L^p Spaces for $0 < p < 1$

Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in (0,1)$. It is easy to check that $L^p(X)$ is a linear space.

Exercise 1. *If* $\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}$ *and* $0 < p < 1$*, then* $\|.\|_p$ *is not a norm on* X*.*

Hint:

Show that $\|.\|_p$ does not satisfy the triangle inequality:

Take $X = [0,1]$ with the Lebesgue measure on it. Let $f = \mathbf{1}_{[0,\frac{1}{2})}$ and $g = \mathbf{1}_{[\frac{1}{2},1)}$. Then show that

$$\|f + g\|_p = 1.$$

and that

$$\|f\|_p = 2^{-\frac{1}{p}} \ \text{ and } \ \|g\|_p = 2^{-\frac{1}{p}}.$$

It follows that

$$\|f + g\|_p > \|f\|_p + \|g\|_p.$$

Exercise 2. *If* $\alpha, \beta \in \mathbb{C}$ *and* $0 < p < 1$*, then*

$$|\alpha + \beta|^p \le |\alpha|^p + |\alpha|^p.$$

Hint:

Consider the real-valued function $\varphi(t) = (1 + t)^p - 1 - t^p, \ t \in [0, \infty)$. Show that it is strictly decreasing on $[0, \infty)$. Then take $t = \frac{|\beta|}{|\alpha|} > 0$.

Exercise 3. *For* $0 < p < 1$, $\|.\|_p$ *is not a norm. However*

$$\rho_p(f,g) := \int_X |f-g|^p d\mu, \quad f,g \in L^p(X)$$

is a metric on $L^p(X)$.

Proof.

We prove only the triangle inequality. For $f, g, h \in L^p(X)$, we have

$$\begin{aligned}
\rho_p(f,g) &= \int_X |f-g|^p d\mu \\
&= \int_X |(f-h)+(h-g)|^p d\mu \\
&\leq \int_X (|f-h|+|h-g|)^p d\mu \\
&\leq \int_X |f-h|^p d\mu + \int_X |h-g|^p d\mu \quad \text{(by Exercise 2)} \\
&= \rho_p(f,h) + \rho_p(h,g). \quad \blacksquare
\end{aligned}$$

* * **

Chapter 11

Integration on Product Measure Space

1. Product measure spaces

Definition 32 *(Product measure)*
Given n measure spaces $(X_1, \mathcal{A}_1, \mu_1), ..., (X_n, \mathcal{A}_n, \mu_n)$. Consider the product measurable space $\big(X_1 \times ... \times X_n, \sigma(\mathcal{A}_1 \times ... \times \mathcal{A}_n)\big)$. A measure μ on $\sigma(\mathcal{A}_1 \times ... \times \mathcal{A}_n)$ such that

$$\mu(E) = \mu_1(A_1)...\mu_n(A_n) \quad \text{for} \quad E = A_1 \times ... \times A_n \in \mathcal{A}_1 \times ... \times \mathcal{A}_n$$

with the convention $\infty.0 = 0$ is called a product measure of $\mu_1, ..., \mu_n$ and we write

$$\mu = \mu_1 \times ... \times \mu_n.$$

Theorem 27 *(Existence and uniqueness)*
For n arbitrary measure spaces $(X_1, \mathcal{A}_1, \mu_1), ..., (X_n, \mathcal{A}_n, \mu_n)$, a product measure space $\big(X_1 \times ... \times X_n, \sigma(\mathcal{A}_1 \times ... \times \mathcal{A}_n), \mu_1 \times ... \times \mu_n\big)$ exists. Moreover, if the n measure spaces are all σ-finite, then the product measure space is unique.

2. Integration

Definition 33 *(Sections and section functions)*
Let $\big(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu\big)$ be the product of two σ-finite measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$. Let $E \subset X \times Y$, and f be an extended real-valued function on E.
(a) For $x \in X$, the set $E(x, .) := \{y \in Y : (x, y) \in E\}$ is called the x-section of E.
For $y \in Y$, the set $E(., y) := \{x \in X : (x, y) \in E\}$ is called the y-section of E.
(b) For $x \in X$, the function $f(x, .)$ defined on $E(x, .)$ is called the x-section of f.
For $y \in Y$, the function $f(., y)$ defined on $E(., y)$ is called the y-section of f.

Proposition 24 *Let $\big(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu\big)$ be the product of two σ-finite measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$. For every $E \in \sigma(\mathcal{A} \times \mathcal{B})$, $\nu\big(E(x, .)\big)$ is a $\mathcal{A}$-measurable function of $x \in X$ and $\mu\big(E(., y)\big)$ is a $\mathcal{B}$-measurable function of $y \in Y$. Furthermore, we have*

$$(\mu \times \nu)(E) = \int_X \nu\big(E(x, .)\big)\mu(dx) = \int_Y \mu\big(E(., y)\big)\nu(dy).$$

Theorem 28 *(Tonelli's Theorem)*
Let $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$ be product measure space of two σ-finite measure spaces. Let f be a non-negative extended real-valued measurable on $X \times Y$. Then

(a) $F^1(x) := \int_Y f(x,.)d\nu$ *is a $\mathcal{A}$-measurable function of $x \in X$.*

(b) $F^2(y) := \int_X f(.,y)d\mu$ *is a $\mathcal{B}$-measurable function of $y \in Y$.*

(c) $\int_{X\times Y} fd(\mu\times\nu) = \int_X F^1 d\mu = \int_Y F^2 d\nu$, *that is,*

$$\int_{X\times Y} fd(\mu\times\nu) = \int_X \left[\int_Y f(x,.)d\nu\right]d\mu = \int_Y \left[\int_X f(.,y)d\mu\right]d\nu.$$

Theorem 29 *(Fubini's Theorem)*
Let $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$ be product measure space of two σ-finite measure spaces. Let f be a $\mu\times\nu$-integrable extended real-valued measurable function on $X \times Y$. Then

(a) *The $\mathcal{B}$-measurable function $f(x,.)$ is ν-integrable on Y for μ-a.e. $x \in X$ and the $\mathcal{A}$-measurable function $f(.,y)$ is μ-integrable on X for ν-a.e. $y \in Y$.*

(b) *The function $F^1(x) := \int_Y f(x,.)d\nu$ is defined for μ-a.e. $x \in X$, $\mathcal{A}$-measurable and μ-integrable on X.*
The function $F^2(y) := \int_X f(.,y)d\nu$ is defined for ν-a.e. $y \in X$, $\mathcal{B}$-measurable and ν-integrable on Y.

(c) *We have the equalities:* $\int_{X\times Y} fd(\mu\times\nu) = \int_X F^1 d\mu = \int_Y F^2 d\nu$, *that is,*

$$\int_{X\times Y} fd(\mu\times\nu) = \int_X \left[\int_Y f(x,.)d\nu\right]d\mu = \int_Y \left[\int_X f(.,y)d\mu\right]d\nu.$$

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Problem 123
Consider the product measure space $\big(\mathbb{R}\times\mathbb{R}, \sigma(\mathcal{B}_\mathbb{R}\times\mathcal{B}_\mathbb{R}), \mu_L\times\mu_L\big)$.
Let $D = \{(x,y)\in\mathbb{R}\times\mathbb{R} : \ x = y\}$. Show that

$$D \in \sigma(\mathcal{B}_\mathbb{R}\times\mathcal{B}_\mathbb{R}) \quad \textit{and} \quad (\mu_L\times\mu_L)(D) = 0.$$

Solution
Let $\lambda = \mu_L \times \mu_L$. Let $D_0 = \{(x,y) \in [0,1]\times[0,1] : x = y\}$. For each $n \in \mathbb{Z}$ let

$D_n = \{(x,y) \in [n, n+1] \times [n, n+1] : x = y\}$. Then, by translation invariance of Lebesgue measure, we have

$$\lambda(D_0) = \lambda(D_n), \ \forall n \in \mathbb{N}.$$
$$\text{and } \ D = \bigcup_{n\in\mathbb{Z}} D_n.$$

To solve the problem, it suffices to prove $D_0 \in \sigma(\mathcal{B}_\mathbb{R} \times \mathcal{B}_\mathbb{R})$ and $\lambda(D_0) = 0$.
For each $n \in \mathbb{N}$, divide $[0,1]$ into 2^n equal subintervals as follows:

$$I_{n,1} = \left[0, \frac{1}{2^n}\right], I_{n,2} = \left[\frac{1}{2^n}, \frac{2}{2^n}\right], ..., I_{n,2^n} = \left[\frac{2^n-1}{2^n}, 1\right].$$

Let $S_n = \bigcup_{k=1}^{2^n} (I_{n,k} \times I_{n,k})$, then $D_0 = \lim_{n\to\infty} S_n$.
Now, for each $n \in \mathbb{N}$ and for $k = 1, 2, ..., 2^n$, $\ I_{n,k} \in \mathcal{B}_\mathbb{R}$. Therefore,

$$I_{n,k} \times I_{n,k} \in \sigma(\mathcal{B}_\mathbb{R} \times \mathcal{B}_\mathbb{R}) \ \text{ and so } \ S_n \in \sigma(\mathcal{B}_\mathbb{R} \times \mathcal{B}_\mathbb{R}).$$

Hence, $D_0 \in \sigma(\mathcal{B}_\mathbb{R} \times \mathcal{B}_\mathbb{R})$.
It is clear that (S_n) is decreasing (make a picture yourself), so

$$D_0 = \lim_{n\to\infty} S_n = \bigcap_{n=1}^{\infty} S_n.$$

And we have

$$\begin{aligned}\lambda(S_n) &= \sum_{k=1}^{2^n} \lambda(I_{n,k} \times I_{n,k}) \\ &= \sum_{k=1}^{2^n} \frac{1}{2^n}.\frac{1}{2^n} = 2^n.\frac{1}{2^{2n}} = \frac{1}{2^n}.\end{aligned}$$

It follows that

$$\lambda(D_0) \leq \lambda(S_n) = \frac{1}{2^n}, \ \forall n \in \mathbb{N}.$$

Thus, $\lambda(D_0) = 0$. ■

Problem 124
Consider the product measure space $(\mathbb{R} \times \mathbb{R}, \sigma(\mathcal{B}_\mathbb{R} \times \mathcal{B}_\mathbb{R}), \mu_L \times \mu_L)$. *Let* f *be a real-valued function of bounded variation on* $[a,b]$. *Consider the graph of* f:

$$G = \{(x,y) \in \mathbb{R} \times \mathbb{R} : \ y = f(x) \ \ for \ \ x \in \mathbb{R}\}.$$

Show that $G \in \sigma(\mathcal{B}_\mathbb{R} \times \mathcal{B}_\mathbb{R})$ *and* $(\mu_L \times \mu_L)(G) = 0$.

Hint:
Partition of $[a, b]$:
$$P = \{a = x_0 < x_1 < ... < x_n = b\}.$$
Elementary rectangles:
$$R_{n,k} = [x_{k-1}, x_k] \times [m_k, M_k], \quad k = 1, ..., n,$$
where
$$m_k = \inf_{x\in[x_{k-1},x_k]} f(x) \quad \text{and} \quad M_k = \sup_{x\in[x_{k-1},x_k]} f(x).$$
Let
$$R_n = \bigcup_{k=1}^{n} R_{n,k} \quad \text{and} \quad \|P\| = \max_{1\le k\le n} (x_k - x_{k-1}).$$
Let $\lambda = \mu_L \times \mu_L$. Show that
$$\lambda(R_n) \le \|P\| \sum_{k=1}^{n} (M_k - m_k) \le \|P\| V_a^b(f),$$

Problem 125
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be the measure spaces given
$$\begin{aligned} &X = Y = [0, 1] \\ &\mathcal{A} = \mathcal{B} = \mathcal{B}_{[0,1]}, \ \textit{the } \sigma\textit{-algebra of the Borel sets in } [0, 1], \\ &\mu = \mu_L \ \textit{and } \nu \textit{ is the counting measure.} \end{aligned}$$
Consider the product measurable space $\big(X \times Y, \sigma(\mathcal{A} \times \mathcal{B})\big)$ and a subset in it defined by $E = \{(x, y) \in X \times Y : \ x = y\}$. Show that
$$\begin{aligned} &(a) \quad E \in \sigma(\mathcal{A} \times \mathcal{B}), \\ &(b) \quad \int_X \left(\int_Y \chi_E d\nu\right) d\mu \ne \int_Y \left(\int_X \chi_E d\mu\right) d\nu. \end{aligned}$$
Why is Tonelli's theorem not applicable?

Solution
(a) For each $n \in \mathbb{N}$, divide $[0, 1]$ into 2^n equal subintervals as follows:
$$I_{n,1} = \left[0, \frac{1}{2^n}\right], I_{n,2} = \left[\frac{1}{2^n}, \frac{2}{2^n}\right], ..., I_{n,2^n} = \left[\frac{2^n - 1}{2^n}, 1\right].$$
Let $S_n = \bigcup_{k=1}^{2^n} (I_{n,k} \times I_{n,k})$. It is clear that (S_n) is decreasing, so
$$E = \lim_{n\to\infty} S_n = \bigcap_{n=1}^{\infty} S_n.$$

Now, for each $n \in \mathbb{N}$ and for $k = 1, 2, ..., 2^n$, $I_{n,k} \in \mathcal{B}_{[0,1]}$. Therefore,

$$I_{n,k} \times I_{n,k} \in \sigma(\mathcal{B}_{[0,1]} \times \mathcal{B}_{[0,1]}) \text{ and so } S_n \in \sigma(\mathcal{B}_{[0,1]} \times \mathcal{B}_{[0,1]}).$$

Hence, $E \in \sigma(\mathcal{B}_{[0,1]} \times \mathcal{B}_{[0,1]})$.
(b) For any $x \in X$, $\mathbf{1}_E(x, .) = \mathbf{1}_{\{x\}}(.)$. Therefore,

$$\int_Y \mathbf{1}_E d\nu = \int_{[0,1]} \mathbf{1}_{\{x\}} d\nu = \nu\{x\} = 1.$$

Hence,

$$\int_X \left(\int_Y \mathbf{1}_E d\nu\right) d\mu = \int_{[0,1]} 1 d\mu = 1. \quad (*)$$

On the other hand, for every $y \in Y$, $\mathbf{1}_E(., y) = \mathbf{1}_{\{y\}}(.)$. Therefore,

$$\int_X \mathbf{1}_E d\mu = \int_{[0,1]} \mathbf{1}_{\{y\}} d\mu = \mu\{y\} = 0.$$

Hence,

$$\int_Y \left(\int_X \mathbf{1}_E d\mu\right) d\nu = \int_{[0,1]} 0 d\mu = 0. \quad (**)$$

Thus, from (*) and (**) we get

$$\int_X \left(\int_Y \mathbf{1}_E d\nu\right) d\mu \neq \int_Y \left(\int_X \mathbf{1}_E d\mu\right) d\nu.$$

Tonelli's theorem requires that the two measures must be σ-finite. Here, the counting measure ν is not σ-finite, so Tonelli's theorem is not applicable. ∎

Question: Why the counting measure on $[0, 1]$ is not σ-finite?

Problem 126
Suppose g is a Lebesgue measurable real-valued function on $[0, 1]$ such that the function $f(x, y) = 2g(x) - 3g(y)$ is Lebesgue integrable over $[0, 1] \times [0, 1]$. Show that g is Lebesgue integrable over $[0, 1]$.

Solution
By Fubini's theorem we have

$$\begin{aligned}
\int_{[0,1]\times[0,1]} f(x,y)d(\mu_L(x)\times\mu_L(y)) &= \int_0^1\int_0^1 f(x,y)dxdy \\
&= \int_0^1\int_0^1 [2g(x)-3g(y)]dxdy \\
&= \int_0^1\int_0^1 2g(x)dxdy - \int_0^1\int_0^1 3g(y)dxdy \\
&= 2\int_0^1 g(x)\left(\int_0^1 1.dy\right)dx - 3\int_0^1 g(y)\left(\int_0^1 1.dx\right)dy \\
&= 2\int_0^1 g(x).1.dx - 3\int_0^1 g(y).1.dy \\
&= 2\int_0^1 g(x)dx - 3\int_0^1 g(y)dy \\
&= -\int_0^1 g(x)dx.
\end{aligned}$$

Since $f(x,y)$ is Lebesgue integrable over $[0,1]\times[0,1]$:

$$\left|\int_{[0,1]\times[0,1]} f(x,y)d(\mu_L(x)\times\mu_L(y))\right| < \infty.$$

Therefore,

$$\left|\int_0^1 g(x)dx\right| < \infty.$$

That is g is Lebesgue (Riemann) integrable over $[0,1]$. ■

Problem 127
Let $(X,\mathfrak{M},\mu)$ be a complete measure space and let f be a non-negative integrable function on X. Let $b(t)=\mu\{x\in X:\ f(x)\geq t\}$. Show that

$$\int_X f d\mu = \int_0^\infty b(t)dt.$$

Solution
Define $F:[0,\infty)\times X\to\mathbb{R}$ by

$$F(t,x)=\begin{cases}1 & \text{if } 0\leq t\leq f(x)\\ 0 & \text{if } t>f(x).\end{cases}$$

If $E_t = \{x \in X : \ f(x) \geq t\}$, then $F(t,x) = \mathbf{1}_{E_t}(x)$. We have

$$\int_0^\infty F(t,x)dt = \int_0^{f(x)} F(t,x)dt + \int_{f(x)}^\infty F(t,x)dt = f(x) + 0 = f(x).$$

By Fubini's theorem we have

$$\begin{aligned}
\int_X f d\mu &= \int_X \left(\int_0^{f(x)} dt \right) dx \\
&= \int_X \left(\int_0^\infty F(t,x)dt \right) dx \\
&= \int_0^\infty \left(\int_X F(t,x)dx \right) dt \\
&= \int_0^\infty \left(\int_X \mathbf{1}_{E_t}(x)dx \right) dt \\
&= \int_0^\infty b(t)dt. \ \ (\text{since} \ \ \mu(E_t) = b(t) \ ). \quad \blacksquare
\end{aligned}$$

Problem 128
Consider the function $u : [0,1] \times [0,1] \to \mathbb{R}$ *defined by*

$$u(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & \textit{for} \ \ (x,y) \neq (0,0), \\ 0 & \textit{for} \ \ (x,y) = (0,0). \end{cases}$$

(a) Calculate

$$\int_0^1 \left(\int_0^1 u(x,y)dy \right) dx \quad \textit{and} \quad \int_0^1 \left(\int_0^1 u(x,y)dx \right) dy.$$

Observation?
(b) Check your observation by using polar coordinates to show that

$$\iint_D |u(x,y)| dxdy = \infty,$$

that is, u *is not integrable. Here* D *is the unit disk.*

Answer.
(a) $\frac{\pi}{4}$ and $-\frac{\pi}{4}$.

Problem 129
Let

$$I[0,1],\ \mathbb{R}_+ = [0,\infty),$$
$$f(u,v) = \frac{1}{1+u^2v^2},$$
$$g(x,y,t) = f(x,t)f(y,t),\ (x,y,t) \in I \times I \times \mathbb{R}_+ := J.$$

(a) Show that g is integrable on J (equipped with Lebesgue measure). Using Tonelli's theorem on $\mathbb{R}_+ \times I \times I$ show that

$$A =: \int_J g dt dx dy = \int_{\mathbb{R}_+} \left(\frac{\arctan t}{t}\right)^2 dt.$$

(b) Using Tonelli's theorem on $I \times I \times \mathbb{R}_+$ show that

$$A = \frac{\pi}{2} \int_{I\times I} \frac{1}{x+y}\, dx dy.$$

(c) Using Tonelli's theorem again show that $A = \pi \ln 2$.

Solution
(a) It is clear that g is continuous on $\mathbb{R}^3$, so measurable. Using Tonelli's theorem on $\mathbb{R}_+ \times I \times I$ we have

$$\begin{aligned}
A &= \int_{\mathbb{R}_+} \left(\int_{I\times I} f(x,t) f(y,t) dx dy \right) dt \\
&= \int_{\mathbb{R}_+} \left(\int_I f(x,t) \left(\int_I f(y,t) dy \right) dx \right) dt \\
&= \int_{\mathbb{R}_+} \left(\left(\int_I \frac{1}{1+x^2t^2}\, dx \right) \left(\int_I \frac{1}{1+y^2t^2}\, dy \right) \right) dt \\
&= \int_{\mathbb{R}_+} \left(\int_I \frac{1}{1+x^2t^2}\, dx \right)^2 dt \\
&= \int_{\mathbb{R}_+} \left(\frac{\arctan t}{t} \right)^2 dt.
\end{aligned}$$

Note that for all $t \in \mathbb{R}_+,\ 0 < \arctan t < \frac{\pi}{2}$ and $\arctan t \sim t$ as $t \to 0$, so

$$A = \int_{\mathbb{R}_+} \left(\frac{\arctan t}{t} \right)^2 dt < \infty.$$

Thus g is integrable on J.

(b) We first decompose $g(x,y,t) = f(x,t)f(y,t)$ into simple elements:

$$\begin{aligned} g(x,y,t) = f(x,t)f(y,t) &= \frac{1}{1+x^2t^2}.\frac{1}{1+y^2t^2} \\ &= \frac{1}{x^2-y^2}\left[\frac{x^2}{1+x^2t^2} - \frac{y^2}{1+y^2t^2}\right]. \end{aligned}$$

Using Tonelli's theorem on $I \times I \times \mathbb{R}_+$ we have

$$\begin{aligned} A &= \int_{I\times I}\left(\int_{\mathbb{R}_+} \frac{1}{x^2-y^2}\left[\frac{x^2}{1+x^2t^2} - \frac{y^2}{1+y^2t^2}\right] dt\right) dxdy \\ &= \int_{I\times I} \frac{1}{x^2-y^2}\left(\int_{\mathbb{R}_+}\left[\frac{x}{1+s^2} - \frac{y}{1+s^2}\right] ds\right) dxdy \\ &= \int_{I\times I} \frac{1}{x+y}\left(\int_0^\infty \frac{ds}{1+s^2}\right) dxdy \\ &= \frac{\pi}{2}\int_{I\times I} \frac{1}{x+y}\ dxdy. \end{aligned}$$

(c) Using (b) and using Tonelli's theorem again we get

$$\begin{aligned} A &= \frac{\pi}{2}\int_0^1 \left(\int_0^1 \frac{1}{x+y}\ dy\right) dx \\ &= \frac{\pi}{2}\int_0^1 [\ln(x+1) - \ln x] dx \\ &= \frac{\pi}{2}[(x+1)\ln(x+1) - x\ln x]_{x=0}^{x=1} = \pi \ln 2. \qquad \blacksquare \end{aligned}$$

Chapter 12

Some More Real Analysis Problems

Problem 130
Let $(X, \mathcal{M}, \mu)$ be a measure space where the measure μ is positive. Consider a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ such that

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

Prove that

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k\right) = 0.$$

Hint:
Let $B_n = \bigcup_{k\geq n} A_k$. Then (B_n) is a decreasing sequence in $\mathcal{M}$ with

$$\mu(B_1) = \sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

Problem 131
Let $(X, \mathcal{M}, \mu)$ be a measure space where the measure μ is positive.
Prove that $(X, \mathcal{M}, \mu)$ is σ-finite if and only if there exists a function $f \in L^1(X)$ and $f(x) > 0,\ \forall x \in X$.

Hint:

- Consider the function

$$f(x)=\sum_{n=1}^{\infty}\frac{\mathbf{1}_{X_n}(x)}{2^n\big[\mu(X_n)+1\big]}.$$

It is clear that $f(x)>0,\ \forall x\in X$. Just show that f is integrable on X.

- Conversely, suppose that there exists $f\in L^1(X)$ and $f(x)>0,\ \forall x\in X$. For every $n\in\mathbb{N}$ set

$$X_n=\Big\{x\in X:\ f(x)>\frac{1}{n+1}\Big\}.$$

Show that

$$\bigcup_{n=1}^{\infty}X_n=X\ \text{ and }\ \mu(X_n)\leq(n+1)\int_X fd\mu.$$

Problem 132

Let $(X,\mathcal{M},\mu)$ be a measure space where the measure μ is positive. Let $f:X\to\overline{\mathbb{R}}_+$ be a measurable function such that $\int_X fd\mu<\infty$.
(a) Let $N=\{x\in X:\ f(x)=\infty\}$. Show that $N\in\mathcal{M}$ and $\mu(N)=0$.
(b) Given any $\varepsilon>0$, show that there exists $\alpha>0$ such that

$$\int_E fd\mu<\varepsilon\ \text{ for any }\ E\in\mathcal{M}\ \text{ with }\ \mu(E)\leq\alpha.$$

Hint:
(a) $N=f^{-1}(\{\infty\})$ and $\{\infty\}$ is closed.
For every $n\in\mathbb{N},\ n\mathbf{1}_N\leq f$.
(b) Write

$$0\leq\int_E fd\mu=\int_{E\cap N^c}fd\mu.$$

For every $n\in\mathbb{N}$ set $g_n:=f\mathbf{1}_{f>n}f\mathbf{1}_{N^c}$. Show that $g_n(x)\to 0$ for all $x\in X$.

Problem 133

Let $\varepsilon>0$ be arbitrary. Construct an open set $\Omega\subset\mathbb{R}$ which is dense in $\mathbb{R}$ and such that $\mu_L(\Omega)<\varepsilon$.

Hint:
Write $\mathbb{Q}=\{x_1,x_2,...\}$. For each $n\in\mathbb{N}$ let

$$I_n:=\Big(x_n-\frac{\varepsilon}{2^{n+2}},x_n+\frac{\varepsilon}{2^{n+2}}\Big).$$

Then the I_n's are open and $\Omega := \bigcup_{n=1}^{\infty} I_n \supset \mathbb{Q}$.

Problem 134
Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose μ is positive and $\mu(X) = 1$ (so $(X, \mathcal{M}, \mu)$ is a probability space). Consider the family

$$\mathcal{T} := \{A \in \mathcal{M} : \ \mu(A) = 0 \text{ or } \mu(A) = 1\}.$$

Show that $\mathcal{T}$ is a σ-algebra.

Hint:
Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{M}$. Let $A = \bigcup_{n\in\mathbb{N}} A_n$.
If $\mu(A) = 0$, then $A \in \mathcal{T}$.
If $\mu(A_{n_0}) = 1$ for some $n_0 \in \mathbb{N}$, then

$$1 = \mu(A_{n_0}) \leq \mu(A) \leq \mu(X) = 1.$$

Problem 135
For every $n \in \mathbb{N}$, consider the functions f_n and g_n defined on $\mathbb{R}$ by

$$f_n(x) = \frac{n^\alpha}{(|x| + n)^\beta} \quad \text{where} \quad \alpha, \beta \in \mathbb{R} \quad \text{and} \quad \beta > 1$$
$$g_n(x) = n^\gamma e^{-n|x|} \quad \text{where} \quad \gamma \in \mathbb{R}.$$

(a) Show that $f_n \in L^p(\mathbb{R})$ and compute $\|f_n\|_p$ for $1 \leq p \leq \infty$.
(b) Show that $g_n \in L^p(\mathbb{R})$ and compute $\|g_n\|_p$ for $1 \leq p \leq \infty$.
(c) Use (a) and (b) to show that, for $1 \leq p < q \leq \infty$, the topologies induced on $L^p \cap L^q$ by L^p and L^q are not comparable.

Hint:
(a)
- For $1 \leq p < \infty$ we have

$$\|f_n\|_p = 2^{\frac{1}{p}} (\beta p - 1)^{-\frac{1}{p}} n^{\alpha - \beta + \frac{1}{p}}.$$

- For $p = \infty$ we have

$$\|f_n\|_\infty = \lim_{p\to\infty} \|f_n\|_p = n^{\alpha-\beta}.$$

(b)
- For $p = \infty$ we have

$$\|g_n\|_\infty = n^\gamma.$$

- For $1 \leq p < \infty$ we have

$$\|g_n\|_p = 2^{\frac{1}{p}} n^{\gamma - \frac{1}{p}} p^{-\frac{1}{p}}.$$

(c) If the topologies induced on $L^p \cap L^q$ by L^p and L^q are comparable, then, for $\varphi_n \in L^p \cap L^q$, we must have

$$(*) \qquad \lim_{n\to\infty} \|\varphi_n\|_p = 0 \Longrightarrow \lim_{n\to\infty} \|\varphi_n\|_q = 0.$$

Find an example which shows that the above assumption is not true. For example:

$$\varphi_n = n^{-\gamma + \frac{1}{q}} g_n.$$

Problem 136
(a) Show that any non-empty open set in $\mathbb{R}^n$ has strictly positive Lebesgue measure.
(b) Is the assertion in (a) true for closed sets in $\mathbb{R}^n$?

Hint:
(a) For any $\varepsilon > 0$, consider the open ball in $\mathbb{R}^n$

$$B_{2\varepsilon}(0) = \{x = (x_1, .., x_n) : \; x_1^2 + ... + x_n^2 < 4\varepsilon^2\}.$$

For each $n \in \mathbb{R}$, let $I_n(0) := \left[-\frac{\varepsilon}{\sqrt{n}}, \frac{\varepsilon}{\sqrt{k}}\right)$. Show that

$$I_\varepsilon(0) := \underbrace{I_n(0) \times ... \times I_n(0)}_{n} \subset B_{2\varepsilon}(0).$$

(b) No.

Problem 137
(a) Construct an open and unbounded set in $\mathbb{R}$ with finite and strictly positive Lebesgue measure.
(b) Construct an open, unbounded and connected set in $\mathbb{R}^2$ with finite and strictly positive Lebesgue measure.
(c) Can we find an open, unbounded and connected set in $\mathbb{R}$ with finite and strictly positive Lebesgue measure?

Hint:
(a) For each $k = 0, 1, 2, ...$ let

$$I_k = \left(k - \frac{1}{2^k}, k + \frac{1}{2^k}\right).$$

Then show that $I = \bigcup_{k=0}^{\infty} I_k$ satisfies the question.
(b) For each $k = 1, 2, ...$ let

$$B_k = \left(-\frac{1}{2^k}, \frac{1}{2^k}\right) \times (-k, k).$$

Then show that $B = \bigcup_{k=0}^{\infty} B_k$ satisfies the question.
(c) No. Why?

Problem 138
Given a measure space $(X, \mathcal{A}, \mu)$. A sequence (f_n) of real-valued measurable functions on a set $D \in \mathcal{A}$ is said to be a Cauchy sequence in measure if given any $\varepsilon > 0$, there is an N such that for all $n, m \geq N$ we have

$$\mu\{x : \ |f_n(x) - f_m(x)| \geq \varepsilon\} < \varepsilon.$$

(a) Show that if $f_n \xrightarrow{\mu} f$ on D, then (f_n) is a Cauchy sequence in measure on D.
(b) Show that if (f_n) is a Cauchy sequence in measure, then there is a function f to which the sequence (f_n) converges in measure.

Hint:
(a) For any $\varepsilon > 0$, there exists $N > 0$ such that for $n, m \geq N$ we have

$$\mu\{D : \ |f_m - f_n| \geq \varepsilon\} \leq \mu\big\{D : \ |f_m - f| \geq \frac{\varepsilon}{2}\big\} + \mu\big\{D : \ |f_n - f| \geq \frac{\varepsilon}{2}\big\}.$$

(b) By definition,

$$\text{for } \delta = \frac{1}{2}, \ \exists n_1 \in \mathbb{N} : \ \mu\Big\{D : \ |f_{n_1+p} - f_{n_1}| \geq \frac{1}{2}\Big\} < \frac{1}{2} \ \text{ for all } p \in \mathbb{N}.$$

In general,

$$\text{for } \delta = \frac{1}{2^k}, \ \exists n_k \in \mathbb{N}, \ n_k > n_{k-1} : \ \mu\Big\{D : \ |f_{n_k+p} - f_{n_k}| \geq \frac{1}{2^k}\Big\} < \frac{1}{2^k} \ \text{ for all } p \in \mathbb{N}.$$

Since $n_{k+1} = n_k + p$ for some $p \in \mathbb{N}$, so we have

$$\mu\Big\{D : \ |f_{n_{k+1}} - f_{n_k}| \geq \frac{1}{2^k}\Big\} < \frac{1}{2^k} \ \text{ for } k \in \mathbb{N}.$$

Let $g_k = f_{n_k}$. Show that (g_k) converges a.e. on D. Let $D_c := \{x \in D : \ \lim_{k\to\infty} g_k(x) \in \mathbb{R}\}$. Define f by $f(x) = \lim_{k\to\infty} g_k(x)$ for $x \in D_c$ and $f(x) = 0$ for $x \in D \setminus D_c$. Then show that $g_k \xrightarrow{\mu} f$ on D. Finally show that $f_n \xrightarrow{\mu} f$ on D.

Problem 139
Check whether the following functions are Lebesgue integrable :
(a) $u(x) = \frac{1}{x},\ x \in [1, \infty)$.
(b) $v(x) = \frac{1}{\sqrt{x}},\ x \in (0, 1]$.

Hint:
(a) $u(x)$ is NOT Lebesgue integrable on $[1, \infty)$.

$$\int_{[1,\infty)} u(x) d\mu_L(x) = \lim_{n\to\infty} \int \frac{1}{x}\ \mathbf{1}_{[1,n)}(x) d\mu_L(x) = \lim_{n\to\infty} \int_1^n \frac{1}{x} dx.$$

(b) $v(x)$ is Lebesgue integrable on $(0, 1]$.
We can write

$$v(x) = \frac{1}{\sqrt{x}},\ x \in (0,1] = \frac{1}{\sqrt{x}}\ \mathbf{1}_{(0,1]}(x) = \sup_n \frac{1}{\sqrt{x}}\ \mathbf{1}_{[\frac{1}{n},1]}(x).$$

Use the Monotone Convergence Theorem for the sequence $\left(\frac{1}{\sqrt{x}}\ \mathbf{1}_{[\frac{1}{n},1]}\right)_{n\in\mathbb{N}}$.

Bibliography

[1] Wheeden, L.R.; Zygmund, A. *Measure and Integral.* Marcel Dekker. New York, 1977.

[2] Folland. G.B. *Real Analysis.* John Wiley and Sons. New York, 1985.

[3] Rudin. W. *Real and Complex Analysis.* Third edition. McGraw-Hill, Inc. New York, 1987.

[4] Royden. H.L. *Real Analysis.* Third edition. Prentice Hall. NJ, 1988.

[5] Rudin. W *Functional Analysis.* McGraw-Hill, Inc. New York, 1991.

[6] Hunter. J.K.; Nachtergaele. B. *Applied Analysis.* World Scientific. NJ, 2001.

[7] Yeh, J. *Real Analysis. Theory of measure and integration.* Second edition. World Scientific. NJ, 2006.